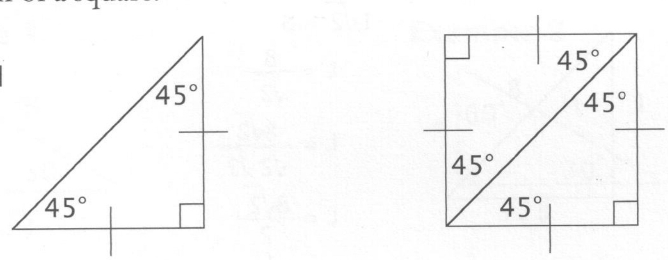
LESSON 20

Special Triangles: 45°- 45°- 90°

45°- 45°- 90° Triangle - Have you noticed the relationship between the legs and the hypotenuse of a 45°- 45 - 90° triangle (right triangle)? Since the angles are congruent, then the opposite sides are also congruent. Think of the triangle shown in figure 1 as half of a square.

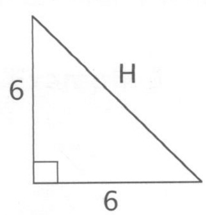
Figure 1



By putting the same number of slashes on the sides, we indicate that they are congruent.

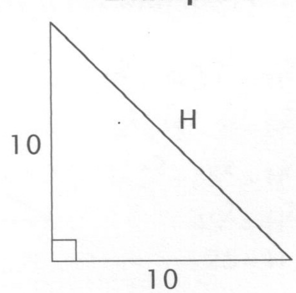
Find the length of the hypotenuse of the following triangles, and observe the common thread.

Example 1



$$6^2+6^2 = H^2$$
$$36+36 = H^2$$
$$72 = H^2$$
$$\sqrt{72} = \sqrt{H^2}$$
$$\sqrt{72} = H$$
$$36\sqrt{2} = H$$
$$6\sqrt{2} = H$$

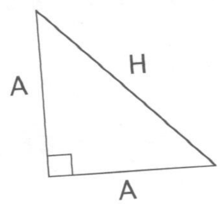
Example 2



$$10^2+10^2 = H^2$$
$$100+100 = H^2$$
$$200 = H^2$$
$$\sqrt{200} = \sqrt{H^2}$$
$$\sqrt{200} = H$$
$$100\sqrt{2} = H$$
$$10\sqrt{2} = H$$

Now let's try a tricky one with A as the length of the legs.

Example 3

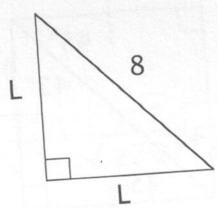


$$A^2 + A^2 = H^2$$
$$2A^2 = H^2$$
$$\sqrt{2A^2} = \sqrt{H^2}$$
$$\sqrt{A^2}\sqrt{2} = H$$
$$A\sqrt{2} = H$$

Rule for Hypotenuse - Have you discovered that the hypotenuse is $\sqrt{2}$ times the leg? If the leg is 5, then the hypotenuse will be $5\sqrt{2}$. If the leg is X, then the hypotenuse is $X\sqrt{2}$.

The rule can also be used to find the length of the legs when the hypotenuse is known.

Example 4



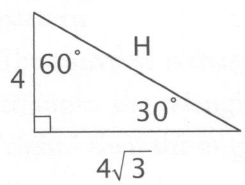
$$L\sqrt{2} = 8$$
$$L = \frac{8}{\sqrt{2}}$$
$$L = \frac{8\sqrt{2}}{\sqrt{2}\sqrt{2}}$$
$$L = \frac{8\sqrt{2}}{2}$$
$$L = 4\sqrt{2}$$

LESSON 21

Special Triangles: 30°–60°–90°

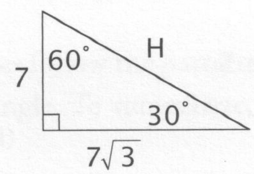
30°–60°–90° Triangle - Another special right triangle is the 30°– 60°–90° right triangle. As in lesson 20, first we'll try to discover the pattern, and then we'll write our formula.

Example 1



$$4^2 + (4\sqrt{3})^2 = H^2$$
$$16 + (16 \cdot 3) = H^2$$
$$16 + 48 = H^2$$
$$\sqrt{64} = \sqrt{H^2}$$
$$8 = H$$

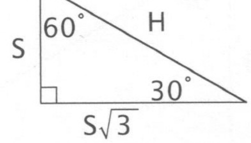
Example 2



$$7^2 + (7\sqrt{3})^2 = H^2$$
$$49 + (49 \cdot 3) = H^2$$
$$49 + 147 = H^2$$
$$196 = H^2$$
$$14 = H$$

Now use your observations in examples 1 and 2 to predict the length of the hypotenuse in example 3, and then work it out with the Pythagorean theorem to confirm your hypothesis.

Example 3

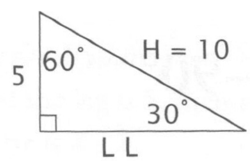


$$S^2 + \left(S\sqrt{3}\right)^2 = H^2$$
$$S^2 + \left(S^2 \cdot 3\right) = H^2$$
$$S^2 + 3S^2 = H^2$$
$$4S^2 = H^2$$
$$2S = H$$

Rule for Hypotenuse - The hypotenuse is twice the length of the short leg (the side opposite the smallest angle). If you are given the short leg, you double it to find the hypotenuse.

Now let's find the relationship between the the short leg (SL) and the long leg (LL). In example 4, we double the short leg to find that the length of the hypotenuse is 10. Our equation to find the long leg is:

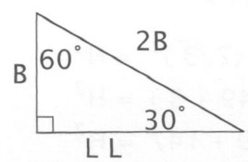
Example 4



$$5^2 + (LL)^2 = 10^2$$
$$25 + (LL)^2 = 100$$
$$(LL)^2 = 75$$
$$LL = \sqrt{25}\sqrt{3}$$
$$LL = 5\sqrt{3}$$

Rule for Short Leg -Now let's use variables to confirm our "guesstimate." If the hypotenuse = 2B, then the short leg = B.

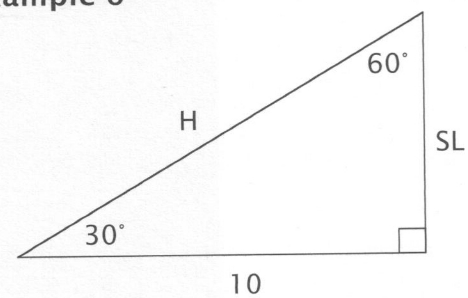
Example 5



$$(B)^2 + (LL)^2 = (2B)^2$$
$$B^2 + (LL)^2 = 4B^2$$
$$(LL)^2 = 3B^2$$
$$LL = \sqrt{B^2}\sqrt{3}$$
$$LL = B\sqrt{3}$$

Rule for Long Leg - The long leg is $\sqrt{3}$ times the short leg. In example 6, find the length of the short leg and the hypotenuse.

Example 6



$$SL\sqrt{3} = LL$$

$$SL\sqrt{3} = 10$$

$$SL\ \frac{\sqrt{3}}{\sqrt{3}} = \frac{10}{\sqrt{3}}$$

$$SL = \frac{10}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

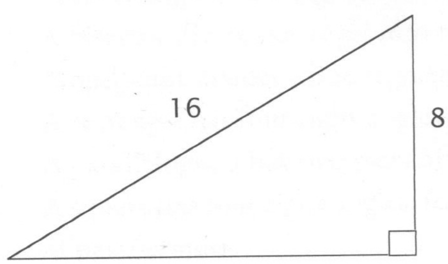
$$H = 2 \times SL$$

$$H = 2 \times \frac{10\sqrt{3}}{3}$$

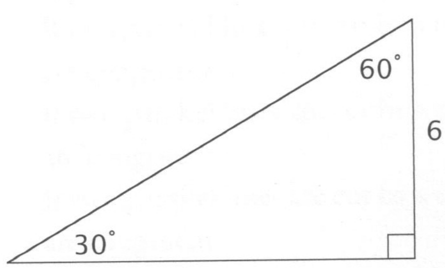
$$H = \frac{20\sqrt{3}}{3}$$

The Converse - If you see that the measures of the angles are 30°–60°–90°, then you know that the length of the sides will follow the pattern of the short leg being one-half of the hypotenuse, and the long leg being the square root of three times the short leg. In short, if the angles are 30°–60°–90°, then the lengths will follow the pattern.

The converse is that if the lengths of the sides follow the pattern of a 30°–60°–90° triangle, the triangle is a 30°–60°–90° triangle. To summarize, if the lengths are "right," then the angles will be "right."



If it is a right triangle and the short leg is one-half the hypotenuse, then you know it is a 30°-60°-90° triangle.



If it is a 30°-60°-90° right triangle, then you know the hypotenuse is two times as long as the short leg (in this case 12).

LESSON 22

Axioms, Postulates, and Theorems

Axioms and *postulates* are observations that one assumes to be true, that make sense, are obvious, but not readily proven. They are defined as unproved assumptions or assertions.

Using axioms or postulates, we make other deductions, and use them to prove new statements true. The result is a *theorem*. Axioms and postulates are assumed, unproved, and obvious. They are used to validate and prove theorems true. Here is a list of observations that we have found to be true so far.

Postulates and Theorems

1. Vertical angles are congruent.
2. A bisector divides an angle into two congruent angles.
3. A midpoint divides a line segment into two congruent segments.
4. A rectangle has four right angles and two pairs of parallel sides.
5. A parallelogram has two pairs of parallel sides.
6. A square has four right angles, four congruent sides, and two pairs of parallel sides.
7. A rhombus has four congruent sides and two pairs of parallel sides.
8. A trapezoid has one pair of parallel sides.
9. If two parallel lines are cut by a transversal, corresponding angles are congruent.
10. If two parallel lines are cut by a transversal, alternate interior angles are congruent.
11. If two parallel lines are cut by a transversal, alternate exterior angles are congruent.
12. Two angles whose measures add up to 180° are supplementary.

13. Two angles whose measures add up to 90° are complementary.
14. If two angles have equal measures, they are congruent.
15. If two line segments have equal lengths, they are congruent.
16. The measures of the interior angles of a triangle add up to 180°.
17. If a triangle has sides A, B, and C, and $A \leq B \leq C$, then $A + B > C$.
18. A regular polygon has all sides congruent and all angles congruent.
19. The measures of the exterior angles of a regular polygon add up to 360°.
20. In a right triangle, leg squared plus leg squared equals hypotenuse squared.
21. An isosceles triangle has two congruent sides.
22. Two lines that intersect and form a right angle are perpendicular.
23. Two lines that are coplanar and do not intersect are parallel.
24. **New:** The property of symmetry: if $A = B$, then $B = A$.
25. **New:** The reflexive property: $A = A$.
26. **New:** The transitive property: if $A = B$ and $B = C$, then $A = C$.

These postulates and theorems are true as are their converses, or inverses.

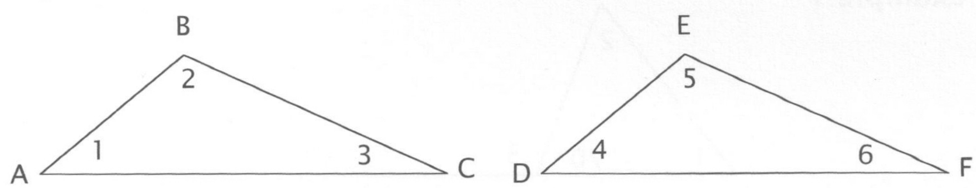
LESSON 23

Corresponding Parts of Triangles

Remote Interior Angles

Congruent Triangles - If you have two triangles that are congruent—that is, all three angles in the first triangle are congruent to all three angles in the second triangle, and all three sides in the first triangle are congruent to all three sides in the second triangle—then it is important how you write this to reflect which angles and sides *correspond*. Observe the two congruent triangles in figure 1.

Figure 1



If you were to place triangle ABC on top of triangle DEF, you would notice that point A corresponds to point D, point B corresponds to point E, and point C corresponds to point F. In the same way, $\angle 1$ corresponds to $\angle 4$, $\angle 2$ corresponds to $\angle 5$, and $\angle 3$ corresponds to $\angle 6$. There are several more statements that may be written, and instead of writing them out, I will use symbols. The arrow means "corresponds to."

$\overline{AB} \leftrightarrow \overline{DE}$ $\qquad$ $\overline{BC} \leftrightarrow \overline{EF}$ $\qquad$ $\overline{CA} \leftrightarrow \overline{FD}$

$\Delta ABC \leftrightarrow \Delta DEF$ $\qquad$ $\Delta ACB \leftrightarrow \Delta DFE$ $\qquad$ $\Delta CBA \leftrightarrow \Delta FED$

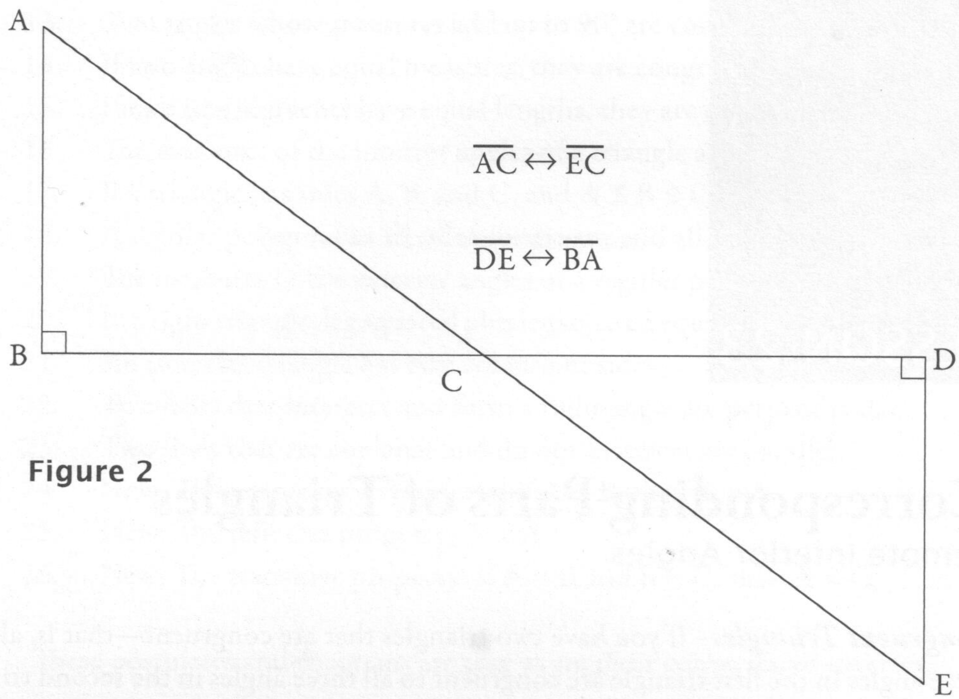
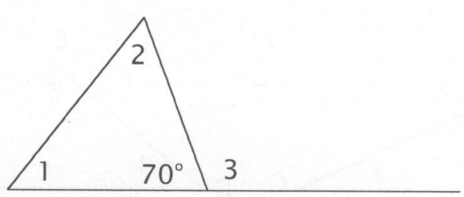


Figure 2

Remote Interior Angles - Note the relationship between ∠1 and ∠2, and ∠3 in example 1 below.

Example 1



The measure of ∠3 must be 110° because it is supplementary to the 70° angle. Now what would ∠1 plus ∠2 equal? We know that the interior angles of a triangle add up to 180°. So ∠1 + ∠2 + 70° = 180°. Study the following equations that show this:

$m\angle 1 + m\angle 2 + 70° = 180°$ Therefore: $m\angle 1 + m\angle 2 = m\angle 3$

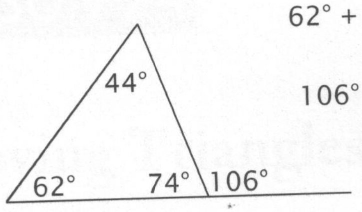
$m\angle 3 + 70° = 180°$ and $m\angle 1 + m\angle 2 = m\angle 3 = 110°$

The transitive property states: if $m\angle 1 + m\angle 2 = 110°$, and $m\angle 3 = 110°$, then $m\angle 1 + m\angle 2 = m\angle 3$.

Read through the section under example 1 several times until you are sure you understand this concept. The exterior angle is $\angle 3$. If you are standing at $\angle 3$, then $\angle 1$ and $\angle 2$ are farthest away from you, and these are called remote interior angles.

Let's look at one more example to make sure this is clear.

Example 2



$62° + 44° + 74° = 180°$

$62° + 44° = 106°$

$106° + 74° = 180°$

We conclude that the measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

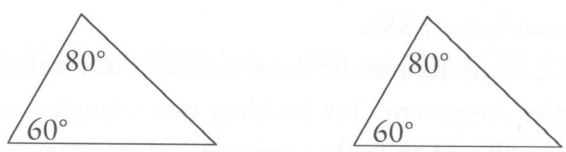
LESSON 24

Proving Triangles Congruent: SSS and SAS

Congruent Triangles - Two triangles are congruent when all three angles and all three sides of one triangle are congruent to the corresponding parts of the second triangle. In order to prove two triangles are exactly the same, we will be using our powers of observation and common sense, as well as all the postulates and theorems recorded in the previous lessons.

What we will be looking for are different combinations of sides and angles that will function as shortcuts. Instead of finding all six corresponding angles and sides the same before we declare two triangles congruent, perhaps we need find only a few. Consider the angles of a triangle. If we know the measures of two of the angles, we really know the measures of all three. Observe figure 1.

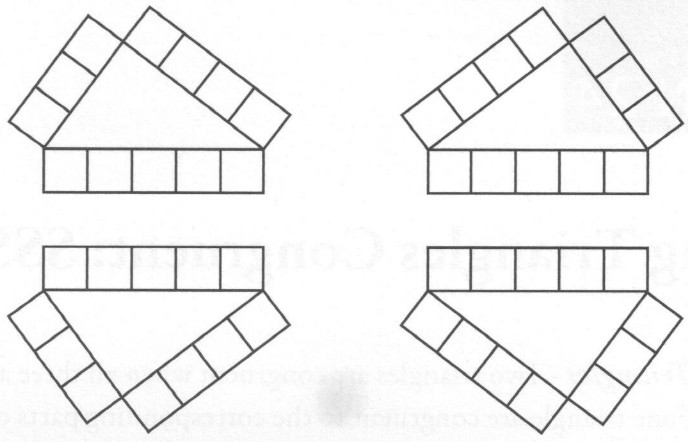
Figure 1



Are all the angles in the first triangle congruent to all the corresponding angles in the second triangle? First find the measures of the two missing angles. We know that the angles in a triangle always add up to 180°, so the missing angles must each be 40°. Both of the triangles are 40°–60°–80°. Earlier, we learned that if we know the measures of two angles, we know the measures of all three, so if two angles of one triangle are congruent to two angles of another triangle, then all three angles are congruent.

Now let's consider the sides of a triangle. Try to build two different triangles with the three sides having lengths of three units, four units, and five units, using the bars. When using the bars, let the inside edges meet at the corners to produce the triangles.

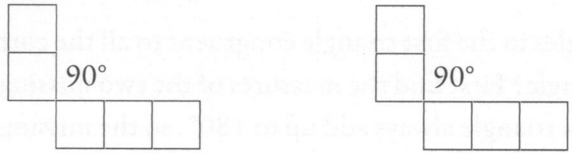
Figure 2



Side-Side-Side Postulate - You may flip the configurations, or twist them in many different shapes, but you will always end up with the same triangle. Another way of saying that the triangles are the same is to say they are congruent. If you were to measure the angles in all the above triangles, you would find the corresponding angles also are the same. What we've found is that if the three corresponding sides are congruent, it follows that the three corresponding angles are congruent as well. Having learned this, we no longer have to check all six measures (all three sides and all three angles) if we have all three sides. We call this the "Side-Side-Side" postulate, or SSS.

There are several combinations of three measures other than SSS that we can use to prove triangles congruent. Try building two triangles, each having a two-bar, a 90° angle, and a three-bar, in that order. Are they congruent?

Figure 3



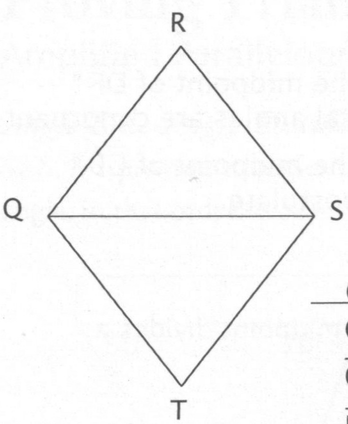
Side-Angle-Side Postulate - There is no other place to put the hypotenuse, is there? What do you think we call this postulate? Since we use a side, then an included angle, and then another side, the postulate is called SAS, for "Side-Angle-Side." Now we'll use these two postulates, SSS and SAS, to prove two triangles congruent.

Proofs - In our proofs, we will always have the following: (1) a figure, or drawing, (2) information given about the figure, and (3) the objective of our proof, or something to prove. As the lawyers in the case, we will do the following: (4) make certain statements, and (5) give reasons to support those statements.

Example 1

(2) **Given:** QRST is a rhombus.

(3) **Prove:** $\Delta QRS \cong QTS$

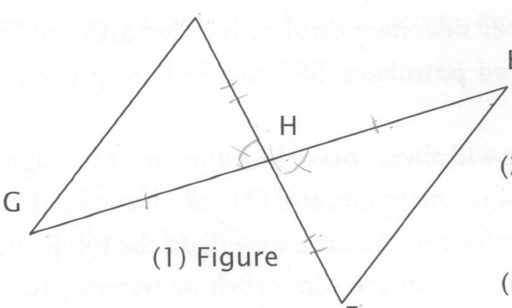


(1) Figure

(4) STATEMENTS	*(5) REASONS*
QRST is a rhombus.	given
$\overline{QR} \cong \overline{QT}$ (side)	definition of a rhombus*
$\overline{RS} \cong \overline{TS}$ (side)	definition of a rhombus*
$\overline{QS} \cong \overline{QS}$ (side)	Reflexive property
$\Delta QRS \cong \Delta QTS$	SSS postulate

*The definition of a rhombus states all the sides are congruent.

Example 2



(1) Figure

(2) **Given:** H is the midpoint of $\overline{DF}$ and $\overline{GE}$.

(3) **Prove:** $\Delta DGH \cong \Delta FEH$

(4) STATEMENTS	*(5) REASONS*
H is the midpoint of $\overline{DF}$ and $\overline{GE}$.	given
$\overline{DH} \cong \overline{FH}$ (S)	H is the midpoint of $\overline{DF}$ *
$\angle DHG \cong \angle FHE$ (A)	Vertical angles are congruent
$\overline{GH} \cong \overline{EH}$ (S)	H is the midpoint of $\overline{GE}$ *
$\Delta DGH \cong \Delta FEH$	SAS postulate

*The definition of a midpoint states that a midpoint divides a segment into two congruent segments.

LESSON 25

Proving Triangles Congruent: ASA and AAS

Amplified Parallelogram Theorem

Angle-Side-Angle Postulate - We have more combinations, besides SSS and SAS, for proving triangles congruent. Try a 50° angle, a violet six-bar, and a 70° angle, in that order.

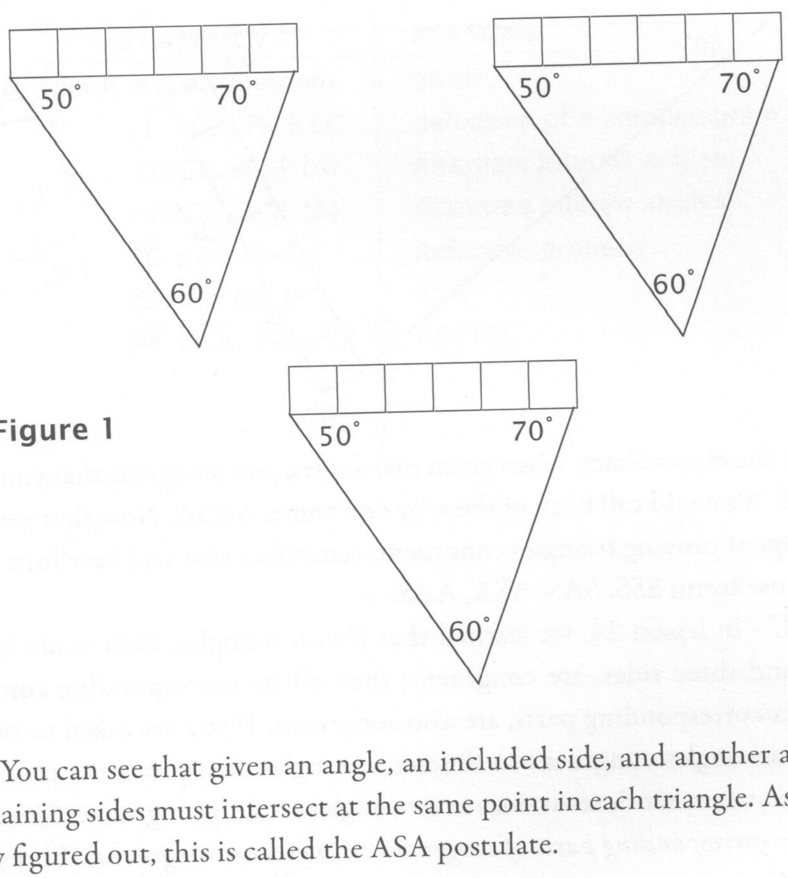


Figure 1

You can see that given an angle, an included side, and another angle, the two remaining sides must intersect at the same point in each triangle. As you've probably figured out, this is called the ASA postulate.

Angle-Angle-Side Postulate - Now try a 40° angle, a 65° angle, and a brown eight bar, in that order. You really don't have to get out your protractor and your blocks to figure out this one. This is simply another way of stating the ASA postulate, because if you are given two angles, you automatically are given all three. In figure 2, if the first two angles are 40° and 65°, then the third angle must be 75°. In figure 1, which illustrates ASA, the angle where the two lines meet must be 60°, right? So ASA leads to the *AAS postulate*.

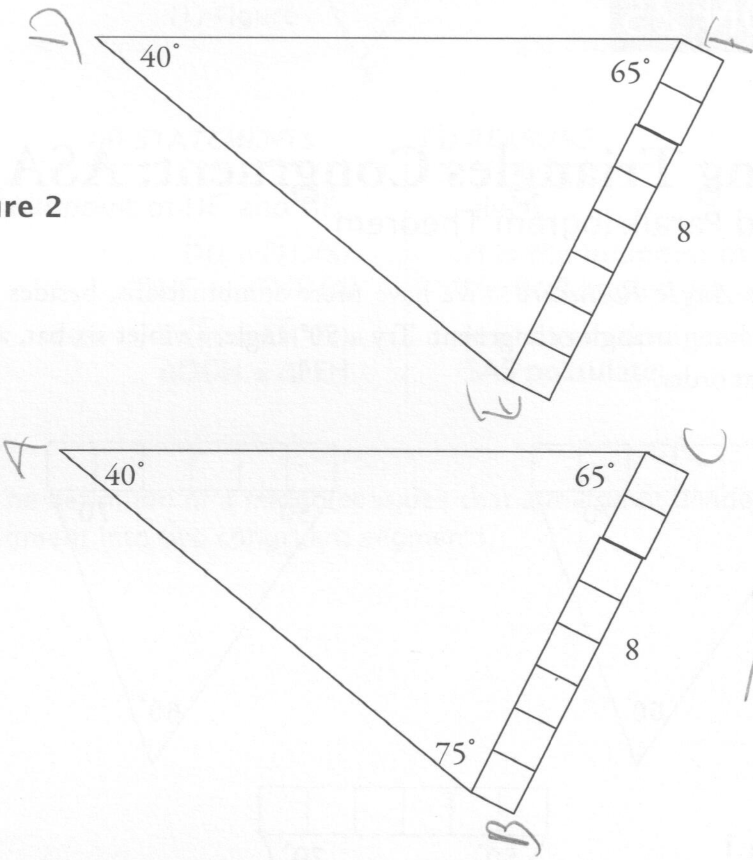


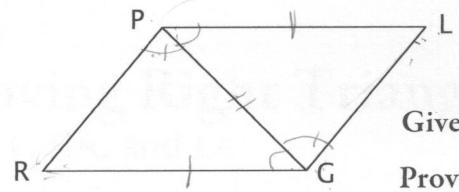
Figure 2

In both of these postulates, when given two angles, you recognize that you also have the third. We could call both of these by one name, AASA. Now that you are becoming adept at proving triangles congruent, remember that you have four postulates to choose from: SSS, SAS, ASA, AAS.

CPCTRC - In lesson 24, we learned that if two triangles, each made up of three angles and three sides, are congruent, then all six corresponding components, or all six corresponding parts, are also congruent. If you are asked to prove two sides or two angles congruent in different triangles, first prove the triangles congruent, then state that the two sides or two angles are also congruent. Give this as your reason: *corresponding parts of congruent triangles are congruent*, abbreviated as *CPCTRC*.

Amplified Parallelogram Theorem (APT) - We have been assuming that opposite sides of a parallelogram are congruent, and you will need to use that fact for some of the proofs in this lesson. Here is a formal proof of the *APT theorem*. If you need to state that two sides of a parallogram or rectangle are congruent, you may use this theorem. Write "opposite sides of a rectangle are congruent" or "opposite side of a parallelogram are congruent" as your reason. You may also simply write APT.

Example 3



Given: ▱ PLGR is a parallelogram.

Prove: $\overline{PR} \cong \overline{GL}$

$\overline{RG} \cong \overline{LP}$

STATEMENTS	***REASONS***
PLGR is a parallegram.	given
$\overline{PL} \parallel \overline{GR}$, $\overline{PR} \parallel \overline{LG}$	definition of a parallelogram
$\angle LPG \cong \angle RGP$ (A)	Alternate interior angles
$\angle RPG \cong \angle LGP$ (A)	Alternate interior angles
$\overline{PG} \cong \overline{PG}$ (S)	Reflexive property
$\Delta PRG \cong \Delta GLP$	ASA
$\overline{PR} \cong \overline{GL}$, $\overline{RG} \cong \overline{LP}$	CPCTRC

LESSON 26

Proving Right Triangles Congruent

HL, LL, HA, and LA

Angles of a Right Triangle - This lesson differs from the two previous lessons in that it applies only to right triangles. The main thing to remember is that of the six components of the two triangles you are seeking to prove congruent, if the triangles are right triangles, one angle in each is already given: the 90° angle. Now you only need one more set of congruent angles to have all three congruent. In a right triangle, if one of the angles is 53°, then you know the other is 37°, because of the 90°, or right angle. With most triangles, you need to know two angles to assure that you have all three congruent. With right triangles, you need only one more angle to assure that you have all three congruent.

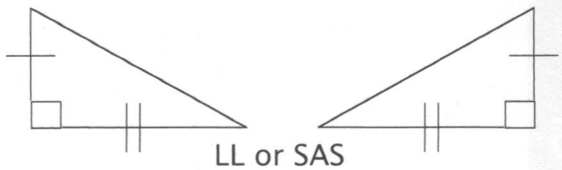
Sides of a Right Triangle - Right triangles are also unique because the Pythagorean theorem tells us if we know the measures of two sides, we also know the length of the third side by computing. If we know the lengths of the two legs, we can square them and find the hypotenuse. If we know the lengths of the hypotenuse and one leg, we can square them, then subtract to find the other leg.

The four new combinations for proving right triangles congruent are identical to SSS, SAS, ASA, and AAS discussed in lessons 24 and 25. The postulates have been renamed using right triangle terminology.

Hypotenuse-Leg (HL) - If you are given the hypotenuse and one leg, you can find the length of the other leg, because of the Pythagorean theorem. So HL (Hypotenuse-Leg) is the same as SSS.

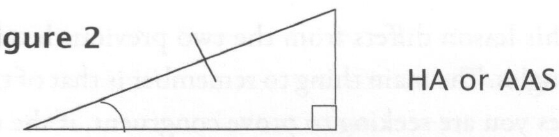
Leg-Leg (LL) - If you are given the two legs, by using the Pythagorean theorem you can find the length of the hypotenuse. So LL (Leg-Leg) is also the same as SSS. But LL could also be SAS because the right angle is between the two legs. We could call it LAL (Leg-Angle-Leg) but it is the same as SAS.

Figure 1



Hypotenuse-Angle (HA) - If you are given the hypotenuse and one angle (not the right angle), or HA, to which of our old postulates is this equivalent? Because you also have the right angle, this is comparable to AAS.

Figure 2



Leg-Angle (LA) - If you are given the leg and one angle (not the right angle), or LA, to which of our old postulates is this equivalent? Because you also have the right angle, this is comparable to either AAS or ASA.

Figure 3

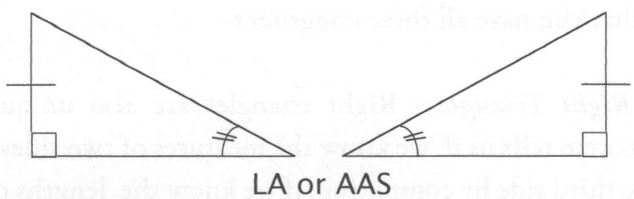
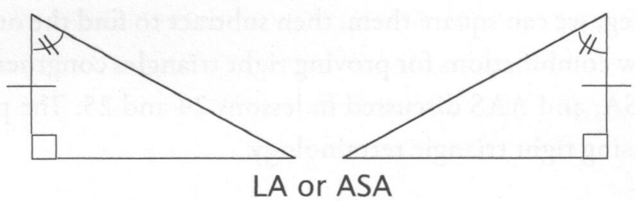


Figure 4



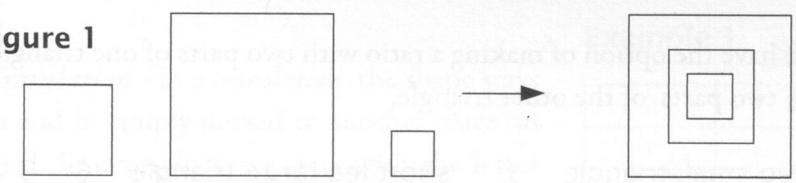
LESSON 27

Proving Triangles Similar with AA

Proportion or Ratio

Definition of Similar - Think of standing beside a smaller (or larger) picture of yourself. You have the same shape as the picture, but you aren't the same size. Or think of a map of your state. The map isn't the same exact size as the state, but it is in the same proportion. Polygons may have the same proportions, or same shape, without being the same size. Two polygons that are not identical in size but have the same proportions are said to be similar (~). Note how the squares in figure 1 are similar but not congruent. The measures of the angles in each square are 90°, but the sides of the squares are different sizes. The squares are similar (~).

Figure 1



In figure 2, we have two triangles that are the same shape and whose corresponding angles have the same measure, but whose corresponding sides are not the same length. (If the angles were the same measure and the sides were the same length, then the triangles would be congruent.)

Figure 2

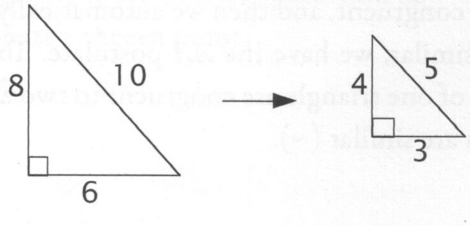
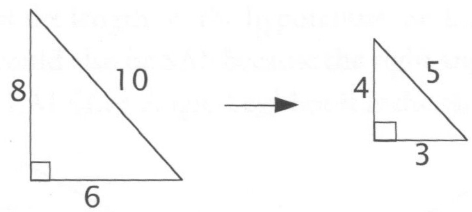


Figure 3



Proportion or Ratio - Figure 3 is the same as figure 2. Looking closely at the corresponding sides, we may write a proportion. You may recall that a proportion is two or more equal ratios. The ratio of the hypotenuse of the small triangle to the hypotenuse of the large triangle is 5 to 10, which may be written as a fraction and subsequently reduced to 1/2.

$$\frac{\text{hyp. small triangle}}{\text{hyp. large triangle}} = \frac{5}{10} = \frac{1}{2}$$

We could also find the ratios of the short legs to the short legs, and the long legs to the long legs, and then put them all together to make a proportion.

$$\frac{\text{short leg small triangle}}{\text{short leg large triangle}} = \frac{3}{6} = \frac{1}{2} \quad \frac{\text{long leg small triangle}}{\text{long leg large triangle}} = \frac{4}{8} = \frac{1}{2}$$

$$\frac{\text{short leg small triangle}}{\text{short leg large triangle}} = \frac{\text{hyp. small triangle}}{\text{hyp. large triangle}} = \frac{\text{long leg small triangle}}{\text{long leg large triangle}} = \frac{3}{6} = \frac{5}{10} = \frac{4}{8}$$

You also have the option of making a ratio with two parts of one triangle proportional to two parts of the other triangle.

$$\frac{\text{short leg small triangle}}{\text{long leg small triangle}} = \frac{3}{4} \quad \frac{\text{short leg large triangle}}{\text{long leg large triangle}} = \frac{6}{8} = \frac{3}{4}$$

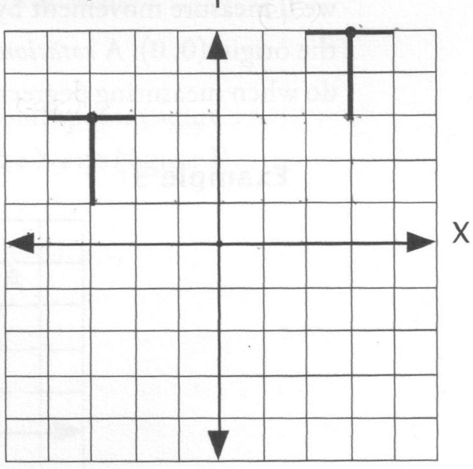
Angle-Angle Postulate (AA) - If the angles of one triangle are congruent to the corresponding angles of another triangle, but the sides aren't the same, the triangles are similar (~). If the triangles are similar, then the corresponding sides will be proportional. In order for all three sets of angles to be congruent, we really only need two sets of angles to be congruent, and then we automatically have the third set. To prove two triangles similar, we have the *AA* postulate. The Angle-Angle postulate states if two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar (~).

LESSON 28

Transformational Geometry

Transformational geometry involves moving geometric shapes around and transforming them on a grid. On a computer, think of drawing a figure on Cartesian coordinates and then doing various commands with your figure. Or, you can pretend you've drawn a shape, and then cut it out and moved it from its original position to another location. What we cover in this lesson are four distinct movements that can be used: translation, reflection, rotation, dilation. The first movement is a translation.

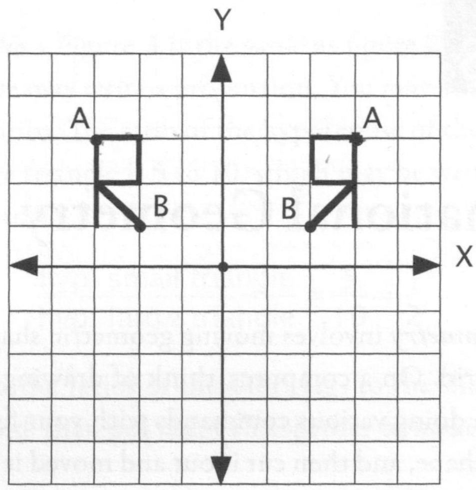
Example 1



Translation - In a *translation*, the shape stays intact and is simply moved to another place on the grid. In example 1, we start with the letter "T" in the second quadrant. The T is moved to the first quadrant. The movement is described in terms of the horizontal (over) and vertical (up or down) coordinates. To measure the movements, pick a point on T. Any point will do. I chose a point at the intersection of the two lines in the letter. On the graph, move over six spaces and up two spaces from the chosen point.

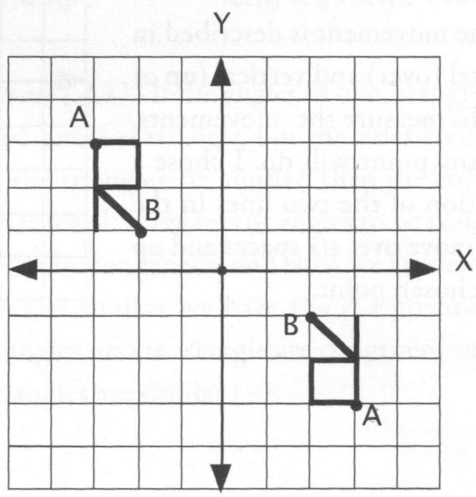
Reflection - Think of a mirror resting on its edge somewhere on the graph. In example 2, we've placed the mirror vertically (running north-south) on the Y-axis. Our figure "R" begins in the second quadrant. I chose two points, A and B, on the R to help in plotting the *reflection* on the graph. The resultant movement is perpendicular to the mirror.

Example 2



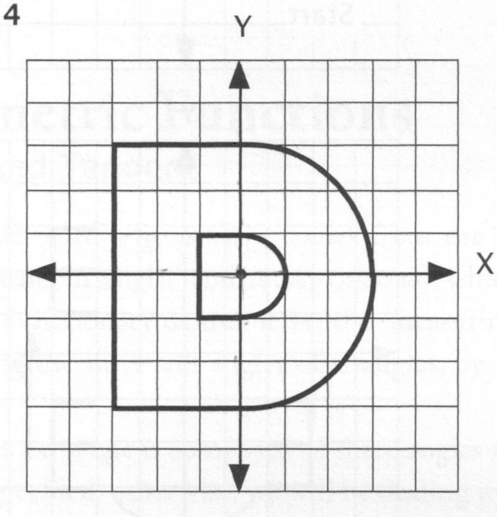
Rotation - When you reflect an object, the central focus is on the location of the mirror. When you rotate an object, the rotation occurs around a specific point. Think of your object lying on the edge of a circle, the center of the circle being the point around which you are moving. Since we are dealing with a circle, we'll measure movement by degrees. In example 3, the R has moved 180° around the origin (0, 0). A *rotation* moves counterclockwise around the circle, just as we do when measuring degrees on a graph.

Example 3



Dilation - When going from darkness into the presence of light, the pupil in the human eye will contract. Conversely, in a dark room, pupils expand (dilate) to allow more light to enter the eye. In the context of transformational geometry, *dilation* is the enlarging or reducing in size of an object without changing its shape. If you have used a computer, you know that you can click and drag on a corner of an object to change its size without changing its shape. In example 4, our shape is a "D" whose edges are one unit from the origin in each direction. We will enlarge it by a factor of three so the resultant D is three times as large in each direction.

Example 4



Combining Transformations - You can also combine transformations. In example 5, E moves from the third quadrant to the first quadrant. There are several possibilities of how it got there:

1. Reflection on the Y-axis and a translation up four spaces (figure 1).
2. Reflection on the X-axis and a reflection on the Y-axis (figure 2).
3. Rotation of 180° around the origin (figure 3).

Example 5

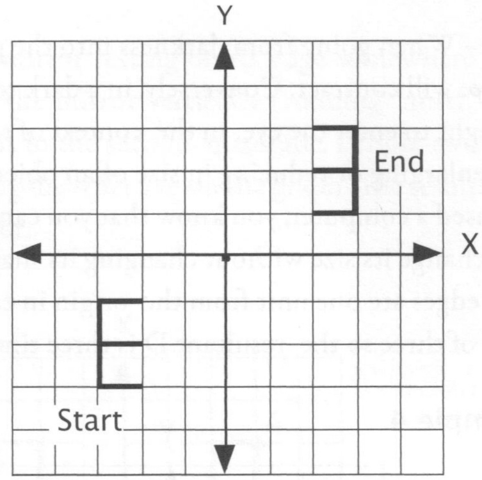


Figure 1

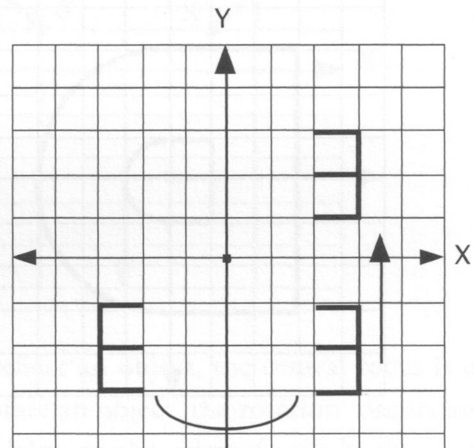


Figure 2

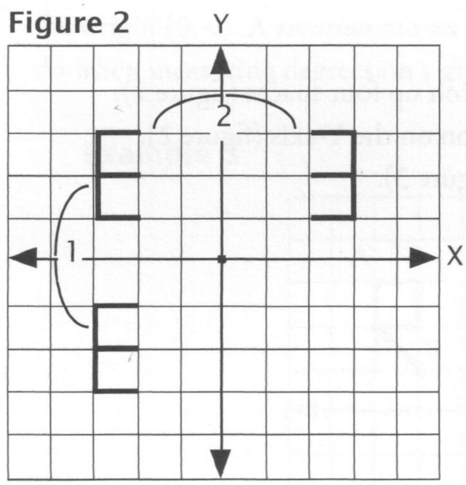
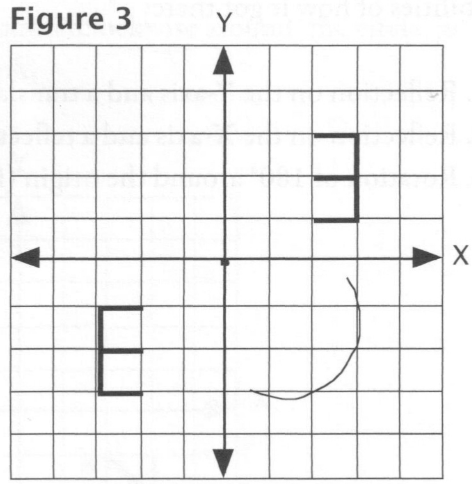


Figure 3



Can you list a different series of transformations?

LESSON 29

Trigonometric Functions

Sine, Cosine and Tangent

Trigonometry - The word *trigonometry* comes from the Greek words τριγονοσ (*trigono*), which means "triangle," and μετρεο (*metreo*), which means "to measure." The Webster's 1828 dictionary defines it as "the measuring of triangles; the science of determining the sides and angles of triangles, by means of certain parts which are given."

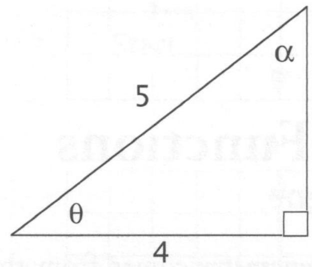
Remember that a triangle is composed of three angles and three sides. In trigonometry, unless specified otherwise, we will be dealing with right triangles. In a right triangle, we already know that one of the angles is 90° and that the side opposite the 90° angle is the hypotenuse. Since we are dealing with right triangles, the Pythagorean theorem also applies, and leg squared plus leg squared equals the hypotenuse squared.

Special right triangles such as the 45°–45°–90° and 30°–60°–90° triangles pertain to our study as well. They are taught in lessons 20 and 21. Before proceeding further, make sure the student understands the material in these two lessons, as they lay the foundation for our study of trigonometry.

In order to measure "the sides and angles of triangles by means of certain parts which are given" as our definition tells us, we need to name the angles and sides. For example, in the 30°–60°–90° right triangle, the three sides are referred to as the short leg, the long leg, and the hypotenuse. Notice that the 30° angle (the smallest angle) is opposite the smallest leg, the 60° angle is opposite the longer leg, and the 90° right angle is opposite the hypotenuse. In the 45°–45°–90° triangle, the two legs are congruent so we can't describe them in terms of comparative length. Let's learn a new system of identification to use in trigonometry.

Describing Sides and Angles - We need to describe all the angles and all the sides in figure 1. Since this is a right triangle, we already know one angle is 90° and the side opposite the right angle is the hypotenuse. This leaves two angles and two sides, or two legs, to name. I've decided to call the angles θ (theta) and α (alpha), both letters in the Greek alphabet. The sides will be described in reference to the angles. If I am standing in angle θ (∠θ), then the leg farthest away from me will be the *opposite* side. The side, or leg, that touches me (where my feet are standing in the illustration) is the *adjacent* side.

Figure 1



I'll illustrate this with our old friend, the 3–4–5 right triangle. Standing at angle θ, the **opposite** side is three units long and the **adjacent** side is four units long. If I move to angle α, then four is **opposite** and three is **adjacent**. In both instances, the hypotenuse is five.

Trigonometric Ratios - Now we come to the six trigonometric ratios that use the terminology of opposite, adjacent, and hypotenuse. In this lesson we will learn the first three ratios, which are *sine* (sin), *cosine* (cos), and *tangent* (tan). The sine of either angle in a right triangle, in our example angle θ or angle α, is described as the ratio of the opposite side over the hypotenuse.

A fun way to remember these three trigonometric ratios is to think of the result of dropping a brick on your big toe. What would you do? Probably get a pan of water and "soak your toe," or SOH-CAH-TOA.

SOH stands for $\underline{s}\text{in} = \frac{\text{opposite}}{\text{hypotenuse}} \Rightarrow S = \frac{O}{H}$

CAH stands for $\underline{c}\text{os} = \frac{\text{adjacent}}{\text{hypotenuse}} \Rightarrow C = \frac{A}{H}$

TOA stands for $\underline{t}\text{an} = \frac{\text{opposite}}{\text{adjacent}} \Rightarrow T = \frac{O}{A}$

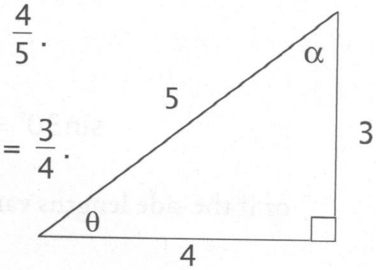
Example 1

Using the 3-4-5 right triangle, the sine of angle θ is $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$

or $\sin\theta = \frac{3}{5}$.

The cosine of θ is the $\frac{\text{adjacent}}{\text{hypotenuse}}$, or $\cos\theta = \frac{4}{5}$.

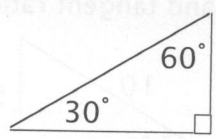
And the tangent of θ is the $\frac{\text{opposite}}{\text{adjacent}}$, or $\tan\theta = \frac{3}{4}$.



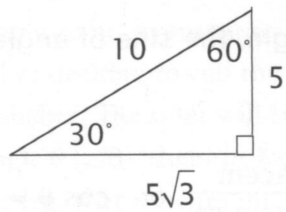
Notice that in a right triangle (which has one right angle), the other two angles always add up to 90°, so they are complementary.

Using the 3–4–5 triangle, let's look at the trigonometric ratios once again. We learned that the ratios of the sides are constant for a 45°–45°–90° triangle or a 30°–60°–90° triangle. In the latter, the short side is always one-half the hypotenuse. The length of the sides of each 30°–60°–90° right triangle may vary, but the short side will always be one-half of the hypotenuse. Using our new terminology we can now say:

$$\sin 30° = \frac{\text{short side}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

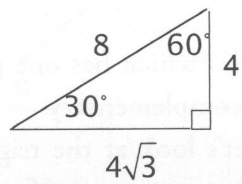


If the sides of the triangle had lengths such as the following:



$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{10} = \frac{1}{2} = .500$$

or if the side lengths varied and you had:

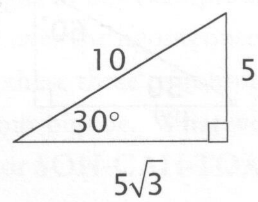


$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{8} = \frac{1}{2} = .500$$

Observe that the ratio of the small side to the hypotenuse remains constant.

Example 1

Find the sine, cosine, and tangent ratios for 30° in the triangle.



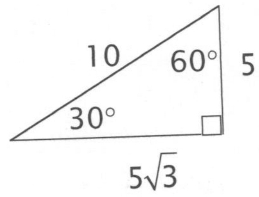
$$\sin 30^\circ = \frac{5}{10} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{5}{5\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 2

Find the sine, cosine, and tangent ratios for 60°
in the same triangle.



$$\sin 60° = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{5}{10} = \frac{1}{2}$$

$$\tan 60° = \frac{5\sqrt{3}}{5} = \sqrt{3}$$

The ratios $\frac{\text{opp}}{\text{hyp}}$, $\frac{\text{adj}}{\text{hyp}}$, and $\frac{\text{opp}}{\text{adj}}$ always depend on what angle is being referred to.

That is why the value of sin 30° and the value of sin 60° are different even though the angles are in the same triangle.

LESSON 30

Inverse Trigonometric Functions

Secant, Cosecant, and Cotangent; $\text{Sin}^2 + \text{Cos}^2 = 1$

Inverse Functions - These are three other ratios that we have in addition to:

$$\sin = \frac{\text{opp}}{\text{hyp}},\ \cos = \frac{\text{adj}}{\text{hyp}},\ \text{and } \tan = \frac{\text{opp}}{\text{adj}}$$

These ratios are the *inverses*, or reciprocals, of our original soh-cah-toa.

Where the sine (sin) $= \dfrac{\text{opp}}{\text{hyp}}$, the *cosecant* (csc) $= \dfrac{\text{hyp}}{\text{opp}}$.

Where the cosine (cos) $= \dfrac{\text{adj}}{\text{hyp}}$, the *secant* (sec) $= \dfrac{\text{hyp}}{\text{adj}}$.

Where tangent (tan) $= \dfrac{\text{opp}}{\text{adj}}$, the *cotangent* (cot) $= \dfrac{\text{adj}}{\text{opp}}$.

Now we have our stable full of all the possible trigonometric ratios. Study the following example to clarify what has been explained.

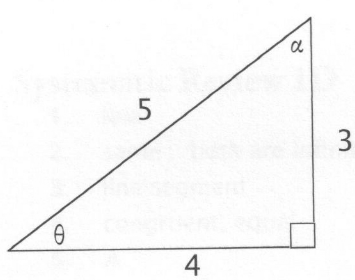
Example 1

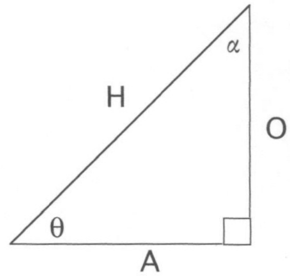
Find all six trig ratios for θ.

$\sin\theta = \dfrac{3}{5}$ $\quad \csc\theta = \dfrac{5}{3}$

$\cos\theta = \dfrac{4}{5}$ $\quad \sec\theta = \dfrac{5}{4}$

$\tan\theta = \dfrac{3}{4}$ $\quad \cot\theta = \dfrac{4}{3}$





$Sin^2\theta + Cos^2\theta = 1$ - Using the triangle above and what we have learned about trigonometric ratios, $\sin\theta = \frac{O}{H}$, and $\cos\theta = \frac{A}{H}$. We also know from the Pythagorean theorem that $O^2 + A^2 = H^2$. Let's put these together to develop an important formula that will be used in the further study of trigonometry.

$O^2 + A^2 = H^2$	Divide through by H^2.
$\frac{O^2}{H^2} + \frac{A^2}{H^2} = 1$	Simplify the ratios.
$\left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2 = 1$	Replace $\frac{O}{H}$ with $\sin\theta$ and $\frac{A}{H}$ with $\cos\theta$.
$(\sin\theta)^2 + (\cos\theta)^2 = 1$	This is the Pythagorean theorem expressed with trig ratios.
$\sin^2\theta + \cos^2\theta = 1$	Here is another way of writing the same thing.