Lesson 9 Cubes and Pascal's Triangle

Before we start cubing binomials, we'll look for a pattern as we square binomials. This will help us square them and factor them more quickly. By now you should be getting the modus operandi under your belt. We'll do several examples of real problems to reveal the pattern, then we'll move to an algebraic equation to formalize our observations and give us a formula. When multiplying these binomials, there are always two options, either distribute them using the FOIL method, or multiply them vertically as you have multiplied numbers for much of your life. I'll mostly do them horizontally with a few vertically, but choose whichever method you are comfortable with.

Example 1

$$(X+4)^2 = (X+4)(X+4) = X^2 + 4X + 4X + 16 = X^2 + 8X + 16$$

Example 2

$$(A+5)^2 = (A+5)(A+5) = A^2 + 5A + 5A + 25 = A^2 + 10A + 25$$

Example 3

$$(Y+7)^2 = (Y+7)(Y+7) = Y^2 + 7Y + 7Y + 49 = Y^2 + 14Y + 49$$

In Examples 4 and 5, the only thing different is the sign.

Example 4

$$(X-3)^2 = (X-3)(X-3) = X^2 - 3X - 3X + 9 = X^2 - 6X + 9$$

Example 5

$$(B-6)^2 = (B-6)(B-6) = B^2 - 6B - 6B + 36 = B^2 - 12B + 36$$

Did you notice the pattern? See if you can predict what $(A+B)^2$ and $(A-B)^2$ will be, then compare your answers with the solutions below.

Formula 1

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + AB + B^2 = A^2 + 2AB + B^2$$

The first term (A) squared <u>plus</u> 2 times the middle term (AB) plus the last term (B) squared.

Formula 2

$$(A-B)^2 = (A-B)(A-B) = A^2 - AB - AB + B^2 = A^2 - 2AB + B^2$$

The first term (A) squared <u>minus</u> 2 times the middle term (AB) plus the last term (B) squared.

Let's use the new formulas to solve the next two examples. Try them first then compare your work. $(X+3)^2$ and $(Y-9)^2$.

Example 6

$$(X+3)^2 = (X)^2 + 2(3X) + (3)^2 = X^2 + 6X + 9$$

Example 7

$$(Y-9)^2 = (Y)^2 + 2(-9Y) + (9)^2 = Y^2 - 18Y + 81$$

The converse is also true. If we see a product with the first and last terms squared and the middle term twice the product of the first and last terms, then we know it is a binomial squared, and we can deduce the factors.

Example 8 Find the binomial root of the trinomial. $X^2 + 14X + 49$

The square root of X^2 is X, and the square root of 49 is 7. Now if the middle term is twice the product of 7X, or 14X, then we are in business. The square root of $X^2 + 14X + 49$ is (X+7).

Example 9 Find the binomial root of the trinomial. $X^2 - 16X + 64$

The square root of X^2 is X, and the square root of 64 is 8. Now if the middle term is twice the product of 8 times X, or 16X, then we are in business. Because it is a negative middle term, then the binomial must be negative. The square root is (Y-8).

Practice Problems Find the product of the binomial squared.

1) $(A+7)^2$	2) $(X-1)^2$	3) $(B+2)^2$	4) $(X-5)^2$
5) $(2A-1)^2$	6) $(2B-3)^2$	7) $(5Y-2)^2$	8) $(3X+4)^2$

Practice Problems Find the square root of the trinomial.

9) $X^2 + 10X + 25$	10) $4A^2 + 12A + 9$	11) $B^2 + 12B + 36$	12) $Y^2 - 12Y + 36$

Solutions

1) $A^2 + 14A + 49$	2) $X^2 - 2X + 1$	3) $B^2 + 4B + 4$	4) $X^2 - 10X + 25$
5) $4A^2 - 4A + 1$	6) $4B^2 - 12B + 9$	7) $25Y^2 - 20Y + 4$	8) $9X^2 + 24X + 16$
9) $(X+5)^2$	10) $(2A+3)^2$	11) $(B+6)^2$	12) $(Y-6)^2$

Cubes Now that you are experts on raising a binomial to a power of two, what about a power of three? Instead of thinking of multiplying (A+B) times itself three times, which can be intimidating, let's multiply (A+B) times (A+B) to the second power, which will give us (A+B) to the third power. When we raise to the third power we often refer to this as "cubed". When we raise to the second power, we say "squared". So (A+B), times (A+B) squared, is (A+B) cubed. We'll do it vertically and horizontally.

Formula 3

$$(A+B)^3 = (A+B)^1 (A+B)^2 = (A+B)^1 (A^2 + 2AB + B^2) = A^3 + 2A^2B + AB^2 + A^2B + 2AB^2 + B^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

Formula 3

$$\begin{array}{r} A^2 + 2AB + B^2 \\ \times \quad A + B \\ \hline BA^2 + 2AB^2 + B^3 \\ A^3 + 2BA^2 + AB^2 \quad\quad \\ \hline A^3 + 3BA^2 + 3AB^2 + B^3 \end{array}$$

Now let's find the formula for (A-B) to the third power.

Formula 4

$$(A-B)^3 = (A-B)^1(A-B)^2 = (A-B)^1 (A^2 - 2AB + B^2) = A^3 - 2A^2B + AB^2 - A^2B + 2AB^2 - B^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

Formula 4

$$
\begin{array}{r}
A^2 - 2AB + B^2 \\
\times \quad A - B \\
\hline
-BA^2 + 2AB^2 - B^3 \\
A^3 - 2BA^2 + AB^2 \quad\quad\quad \\
\hline
A^3 - 3BA^2 + 3AB^2 - B^3
\end{array}
$$

Example 10 Solve $(X+5)^3$, by multiplying $(X+5)(X+5)^2$ and then using the formula.

$(X+5)^3 = (X+5)(X+5)^2 = (X+5)(X^2 +10X + 25) = X^3 +10X^2 + 25X + 5X^2 + 50X + 125 = X^3 +15X^2 + 75X + 125$

$(X+5)^3 = (X)^3 + 3(X)^2 (5) + 3(X)(5)^2 + (5)^3 = X^3 + 15X^2 + 75X + 125$

Practice Problems

1) $(A + 2)^3$ 2) $(X + 10)^3$ 3) $(2A + 4)^3$ 4) $(2X + 2Y)^3$

5) $(B - 2)^3$ 6) $(Y - 10)^3$ 7) $(3B - 1)^3$ 8) $(R - 1/2)^3$

Solutions

1) $(A+2)^3 = (A)^3 + 3(A)^2 (2) + 3(A)(2)^2 + (2)^3 = A^3 + 6A^2 + 12A + 8$

2) $(X+10)^3 = (X)^3 + 3(X)^2 (10) + 3(X)(10)^2 + (10)^3 = X^3 + 30X^2 + 300X + 1,000$

3) $(2A+4)^3 = (2A)^3 + 3(2A)^2 (4) + 3(2A)(4)^2 + (4)^3 = 8A^3 + 48A^2 + 96A + 64$

4) $(2X+2Y)^3 = (2X)^3 + 3(2X)^2 (2Y) + 3(2X)(2Y)^2 + (2Y)^3 = 8X^3 + 24X^2Y + 24XY^2 + 8Y^3$

5) $(B-2)^3 = (B)^3 - 3(B)^2 (2) + 3(B)(2)^2 - (2)^3 = B^3 - 6B^2 + 12B - 8$

6) $(Y-10)^3 = (Y)^3 - 3(Y)^2 (10) + 3(Y)(10)^2 - (10)^3 = Y^3 - 30Y^2 + 300Y - 1,000$

7) $(3B-1)^3 = (3B)^3 - 3(3B)^2 (1) + 3(3B)(1)^2 - (1)^3 = 27B^3 - 27B^2 + 9B - 1$

8) $(R-1/2)^3 = (R)^3 - 3(R)^2 (1/2) + 3(R)(1/2)^2 - (1/2)^3 = R^3 - 3/2\ R^2 + 3/4\ R - 1/8$

Pascal's Triangle We know how to raise a binomial to a power of 0, 1, 2, and 3. We can keep going, and with a good bit of ink, figure out how to raise it to the 4th power and on up. But a pattern has been emerging in the coefficients as well as the variables. Next we'll raise (A+B) to the power of 4, then review and look for the pattern. A young man named Blaise Pascal is credited with discovering this pattern. Let's follow in his footsteps.

Example 11

$$
\begin{array}{r}
A^3 + 3A^2B + 3AB^2 + B^3 \\
\times \quad A + B \\
\hline
A^3B + 3A^2B^2 + 3AB^3 + B^4 \\
A^4 + 3A^3B + 3A^2B^2 + AB^3 \quad\quad\; \\
\hline
A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4
\end{array}
$$

Notice the coefficients in the following sequence.

coefficients, variables & exponents

$(A+B)^0$	1
$(A+B)^1$	$1A^1 + 1B^1$
$(A+B)^2$	$1A^2 + 2A^1B^1 + 1B^2$
$(A+B)^3$	$1A^3 + 3A^2B^1 + 3A^1B^2 + 1B^3$
$(A+B)^4$	$1A^4 + 4A^3B^1 + 6A^2B^2 + 4A^1B^3 + 1B^4$

coefficients & variables

$(A+B)^0$	1
$(A+B)^1$	1A +1B
$(A+B)^2$	1A + 2A B + 1B
$(A+B)^3$	1A + 3A B + 3A B + 1B
$(A+B)^4$	1A + 4A B + 6A B + 4A B + 1B

coefficients

$(A+B)^0$	1
$(A+B)^1$	1 1
$(A+B)^2$	1 2 1
$(A+B)^3$	1 3 3 1
$(A+B)^4$	1 4 6 4 1

While studying the triangle with just the coefficients, see if you notice the pattern, and then try and predict what the coefficients would be for (A+B) to the fifth power. After you've done this, read the following explanation.

coefficients

1

1 1 — Add (1+1) in line two, to get 2 in line three.

1 2 1 — Add (1+2) in line three, to get 3 in line four.

1 3 3 1 — Add (1+3) in line four, to get 4 in line five and (3+3) to get 6.

1 4 6 4 1

1 5 10 10 5 1 — This is what the next line should be.

Practice Problems *Find the coefficients for the next three rows representing the 6th, 7th, and 8th power.*

Solutions

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1