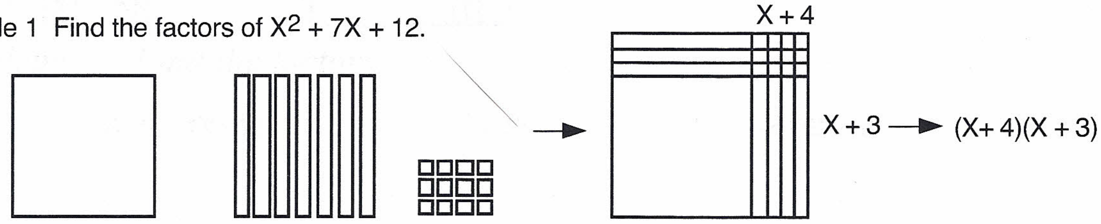
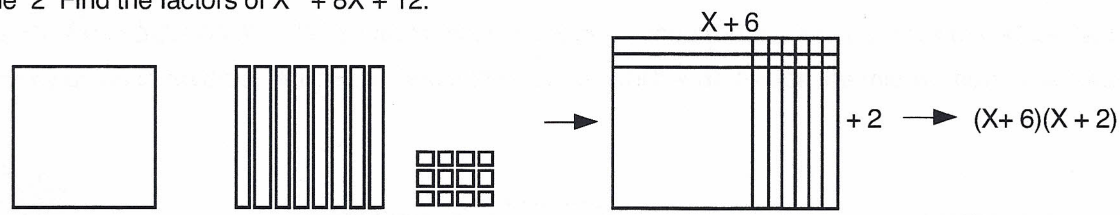
Lesson 5 Factoring Polynomials and More Rational Expressions

Factoring Trinomials with Positive Numbers We will be finding the factors of $X^2 + 7X + 12$ using the blocks with the algebra inserts snapped into the back. First build $X^2 + 7X + 12$. This is the product which is given. Now build a rectangle using all the blocks. Then find the factors by reading the length of the "over" dimension and the "up" dimension.

Example 1 Find the factors of $X^2 + 7X + 12$.



Example 2 Find the factors of $X^2 + 8X + 12$.



Notice the relationship between the last term (12), the middle term (7X or 8X), and the factors.

$X^2 + 7X + 12 =$ (X+ 4)(X + 3) The last term is found by <u>multiplying</u> 3 x 4. The middle term by <u>adding</u> 3X + 4X.

$X^2 + 8X + 12 =$ (X+ 6)(X + 2) The last term is found by multiplying 6 x 2. The middle term by adding 6X + 2X.

Summary: If the coefficient of X^2 is 1, then the factors of the last term are the addends of the middle term.

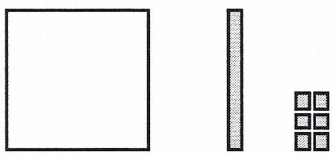
Practice Problems *Find the factors.*

1) $X^2 + 5X + 6$ 2) $X^2 + 13X + 12$ 3) $X^2 + 13X + 42$ 4) $X^2 + 9X + 20$ 5) $X^2 + 6X + 9$

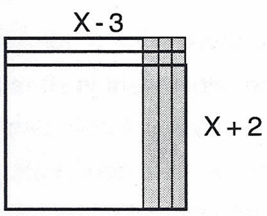
Solutions

1) (X + 2)(X + 3) 2) (X + 12)(X + 1) 3) (X + 6)(X + 7) 4) (X + 4)(X + 5) 5) (X + 3)(X + 3)

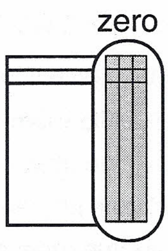
Factoring Trinomials with Negative Numbers The shaded bars are negative X (-X). They are represented by gray algebra inserts snapped into the back of the blue ten bars. The shaded unit bars are unit pieces upside down (the hollow side is showing) representing negative numbers. $X^2 - X - 6$ looks like this:



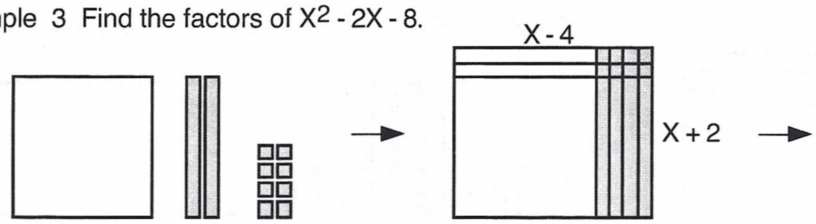
To find the factors, we must build a rectangle. But this is not possible with these manipulatives, so we have to add something to this. We know that zero plus anything is still the same thing. So, we will add zero (nothing) to this by adding a positive X and a negative X, or a gray insert and a blue insert. This still doesn't allow us enough pieces to build a rectangle, so we'll add zero again (+X-X). This works! We still have $X^2 - X - 6$ but now it is in a different form: $X^2 - 3X + 2X - 6$. The factors are (X + -3)(X + 2) or (X - 3)(X + 2).



For clarity, pick up all the negative pieces and place them on top of the positive pieces. Think of the negative pieces and the positive pieces directly beneath them as making zero. See the factors in the remaining rectangle.



Example 3 Find the factors of $X^2 - 2X - 8$.



→ $(X-4)(X+2)$

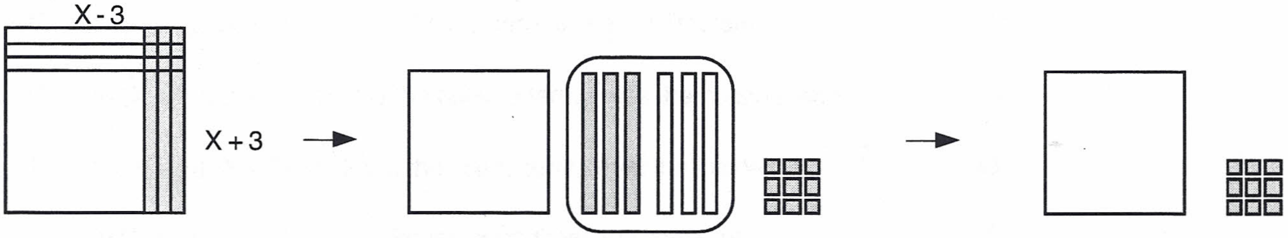
Practice Problems *Find the factors.*

1) $X^2 + 2X - 15$ 2) $X^2 - 4X - 12$ 3) $X^2 + 6X - 7$ 4) $X^2 - 6X + 8$ 5) $X^2 - 9X + 20$

Solutions

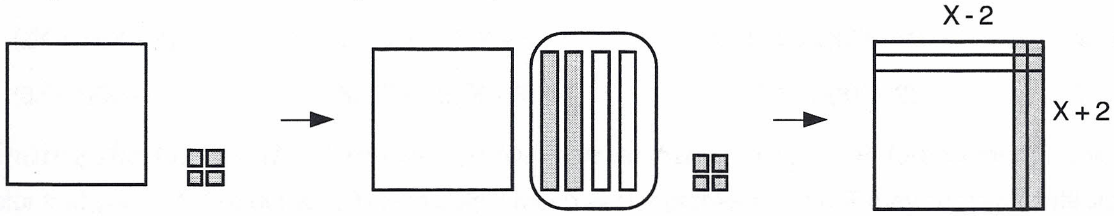
1) $(X - 3)(X + 5)$ 2) $(X + 2)(X - 6)$ 3) $(X + 7)(X - 1)$ 4) $(X - 2)(X - 4)$ 5) $(X - 4)(X - 5)$

The Difference of Two Squares When factoring, we are given the product, and we have to find the factors. Let's begin this type by going in reverse. Build a rectangle to solve $(X - 3)(X + 3)$. Notice the middle term especially. It is equal to zero.



The first term X^2 is a square, as is the third term $(-3)^2$, which is 9. So when asked to factor the "difference of two squares", the answer is the square root of the first term <u>plus</u> the square root of the third term, times the square root of the first term <u>minus</u> the square root of the third term. Algebraically we may represent this pattern as $A^2 - B^2 = (A + B)(A - B)$. In the problem we solved, this is $X^2 - 9 = (X+3)(X-3)$.

Example 4 *Find the factors of $X^2 - 4$.*



$X^2 - 4 = (X+2)(X-2)$

Practice Problems *Find the factors.*

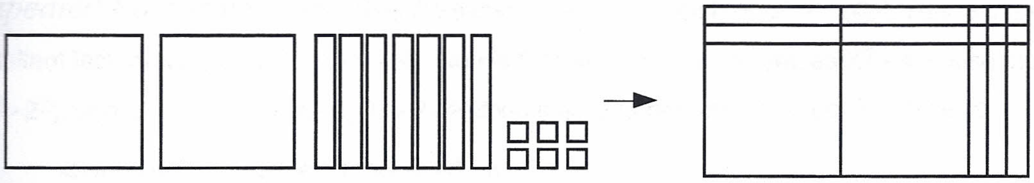
1) $X^2 - 36$ 2) $X^2 - A^2$ 3) $X^2 - 1$ 4) $4X^2 - 9$ 5) $9X^2 - 25$

Solutions

1) $(X - 6)(X + 6)$ 2) $(X + A)(X - A)$ 3) $(X + 1)(X - 1)$ 4) $(2X - 3)(2X + 3)$ 5) $(3X - 5)(3X + 5)$

Factoring Trinomials with Coefficients In a polynomial, coefficients are the numbers, and variables (X^3, X^2, X^1, etc.) are the place values. The coefficient tells "how many", and the variable tells "what kind". In $2X^2$, the 2 is the coefficient. In 7X, the 7 is the coefficient. When factoring a trinomial, when the coefficient of X^2 is 1, the factors of the last term are the addends of the middle term. But when the coefficient is not 1, then the factoring process becomes more difficult. Now instead of finding the factors of the third term alone, you also have factors of the first term to contend with. And if the sign is negative, then there are more options. You can always work out all the possibilities systematically until you find the solution, which can be tedious. After a lot of practice you begin to see the patterns which reveal the solutions more quickly.

Example 5 Find the factors of $2X^2 + 7X + 6$.



$2X^2 + 7X + 6 = (2X+3)(X+2)$

$(2X + 3)(X + 2)$ may be multiplied to check the factors vertically (below left), or linearly, using the distributive property.

$$\begin{array}{r} 2X + 3 \\ X + 2 \\ \hline 4X + 6 \\ 2X^2 + 3X \quad \\ \hline 2X^2 + 7X + 6 \end{array}$$

$(X + 2)(2X + 3) = (X)(2X+ 3) + (2)(2X+ 3) = 2X^2 + 3X + 4X + 6$

Another way that factoring has been taught is using the foil method. FOIL represents the four products, or combinations, of the factors: F - first, O - outside, I - inside, L - last.

F	In $(X + 2)(2X + 3)$, $X \cdot 2X$ is the first term times the first term	$2X^2$
O	In $(X + 2)(2X + 3)$, $X \cdot 3$ is the outside term times the outside term	$3X$
I	In $(X + 2)(2X + 3)$, $2 \cdot 2X$ is the inside term times the inside term	$4X$
L	In $(X + 2)(2X + 3)$, $2 \cdot 3$ is the last term times the last term	6

Practice Problems *Find the factors.*

1) $2X^2 + 9X + 4$	2) $3X^2 + 17X + 10$	3) $4X^2 + 8X + 3$	4) $6X^2 + 19X + 15$
5) $3X^2 - X - 4$	6) $5X^2 + 13X - 6$	7) $6X^2 + X - 2$	8) $12X^2 - 11X + 2$

Solutions

1) $(2X + 1)(X + 4)$	2) $(3X + 2)(X + 5)$	3) $(2X + 3)(2X + 1)$	4) $(2X + 3)(3X + 5)$
5) $(3X - 4)(X + 1)$	6) $(5X - 2)(X + 3)$	7) $(2X - 1)(3X + 2)$	8) $(3X - 2)(4X - 1)$

Finding the Greatest Common Factor before Factoring Before factoring, look for a greatest common factor that you can take out first. This makes a much easier problem. A GCF may contain coefficients, variables, or both.

Example 6 Find the factors of $2X^2 - 18$.

Notice that both have a common factor of 2, so we can make this $2(X^2 - 9)$. $(X^2 - 9)$ is the difference of two squares, $X^2 - 3^2$, so our factors are $2(X+3)(X-3)$.

Example 7 Find the factors of $3X^3 + 9X^2 - 54X$.

Notice that each term has a common factor of 3. Each term also has a common factor of X. We can factor 3X out of each term, yielding $3X(X^2 + 3X - 18)$. Then we can factor. The final result is: $3X(X + 6)(X - 3)$.

Practice Problems *Find the GCF and then the factors.*

1) $2X^3 + 12X^2 + 18X$	2) $3X^2 - 9X + 6$	3) $4X^4 - 25X^2$	4) $20X^4 + 10X^3 - 30X^2$

Solutions

1) $2X(X + 3)(X + 3)$	2) $3(X - 1)(X - 2)$	3) $X^2(2X + 5)(2X - 5)$	4) $10X^2(2X + 3)(X - 1)$

Repeated Factoring $X^4 - 16$ is the same as $(X^2)(X^2) - (4)(4)$ or $(X^2)^2 - (4)^2$. This is the difference of two squares. The resultant factors will be $(X^2 - 4)(X^2 + 4)$. Notice that we are not done yet, as $X^2 - 4$ is also the difference of two squares, $(X^2 - 2^2)$, and its factors are $(X - 2)(X + 2)$. Putting it all together: $X^4 - 16 = (X^2)^2 - (4)^2 = (X^2 + 4)(X^2 - 4) = (X^2 + 4)(X - 2)(X + 2)$

Example 8 Find the factors of $X^4 - 81$.

$$X^4 - 81 = (X^2)^2 - (9)^2 = (X^2 + 9)(X^2 - 9) = (X^2 + 9)(X - 3)(X + 3)$$

Practice Problems *Find the factors.*

1) $X^4 - 1$

2) $X^4 - 104X^2 + 400$

Solutions

1) $(X^2 + 1)(X^2 - 1) = (X^2 + 1)(X+1)(X-1)$

2) $(X^2 - 4)(X^2 - 100) = (X+2)(X-2)(X+10)(X-10)$

Now after learning all of these skills about factoring, you can apply them to solve an equation.

Using Factoring to Solve Equations

Example 9 Find the solution of $X^2 + 5X + 6 = 20$

$X^2 + 5X + 6 = 20$ First subtract 20 from both sides.

$X^2 + 5X - 14 = 0$ Now find the factors.

$(X + 7)(X - 2) = 0$ Think of these two factors as A & B.

If A x B = 0, then either A, or B, or both, have to be equal to zero.

The options are: 0 x B = 0, or A x 0 = 0, or 0 x 0 = 0.

So either $(X + 7) = 0$ or $(X - 2) = 0$. If $X + 7 = 0$ then $X = -7$, If $X - 2 = 0$ then $X = 2$.

The solutions then are $X = -7$ or 2. Let's check these in the original equation.

$(-7)^2 + 5(-7) + 6 = 20$ and $(2)^2 + 5(2) + 6 = 20$.

$49 - 35 + 6 = 20$ $4 + 10 + 6 = 20$

They both work, thus validating our solutions.

Practice Problems Find the solution(s) for X. In other words, "What values of X will satisfy the equation?" Then check your answers by substituting them into the original equation to see if they work.

1) $2X^2 + 3X = 2$

2) $3X^3 = 27X$

Solutions

1) A.	$2X^2 + 3X - 2 = 0$			A. Set the equation equal to zero.
B.	$(2X - 1)(X + 2) = 0$	$2X - 1 = 0$	$X + 2 = 0$	B. Find the factors and set them equal to zero.
C.		$X = 1/2$	$X = -2$	C. Solve the equations.
D.		$2(1/2)^2 + 3(1/2) - 2 = 0$	$2(-2)^2 + 3(-2) - 2 = 0$	D. Check by substituting the solutions.
		$0 = 0$	$0 = 0$	
		Yes	Yes	

2) A	$3X^3 - 27X = 0$				A. Set the equation equal to zero.
B.	$3X(X + 3)(X - 3)$	$3X = 0$	$X+3=0$	$X-3=0$	B. Find the factors and set them equal to zero.
C.		$X = 0$	$X = -3$	$X = 3$	C. Solve the equations.
D.		$3(0)^3 - 27(0) = 0$	$3(-3)^3 - 27(-3) = 0$	$3(3)^3 - 27(3) = 0$	D. Check by substituting the solutions.
		$0 = 0$	$0 = 0$	$0 = 0$	
		Yes	Yes	Yes	

More Rational Expressions When we put what we know about polynomials with what we've been learning about rational expressions ,we will have some interesting equations, like puzzles, to solve. But in the process, remember that you can never have zero as a denominator. The denominator is the divider, or divisor, and you can't divide by zero.

If $\frac{6}{2} = 3$ Then 2 x 3 = 6. This is true. If $\frac{18}{X} = 9$ Then X = 2, because 2 x 9 = 18.

What if you were to solve this equation? What is X? Or, what times 0 equals 18? $\frac{18}{X} = 0$

This is the same equation. What is X here? Or, what times 0 equals 18? $\frac{18}{0} = X$

There is no such solution to either of these, and so we say that the solutions are undefined.

This applies to polynomials because you often have variables which are in the denominator. Look at the examples and notice the denominator. What are the two values that X cannot be?
X cannot be1 and -3, because then the denominator would be zero, and the solution would be undefined.

$$\frac{X}{X-1} + \frac{X}{X+3} = 5 \qquad \frac{X}{1-1} + \frac{X}{-3+3} = 5 \qquad \frac{X}{0} + \frac{X}{0} = 5$$

When we have possibilities for X to be zero, we qualify the answer by saying: $X \neq 1$ or $X \neq -3$, as in our example. This states that X can be any number <u>except</u> 1 or -3.

Keeping this in mind, let's solve some more difficult equations with rational expressions. Combine the following expressions in this example.

Example 1 Combine.

The common denominator is (X+5)(X-5) which is X^2 -25, and $X \neq 5, -5$

$$\frac{2}{X+5} + \frac{5}{X-5} - \frac{10}{X^2-25} \longrightarrow \frac{2}{X+5}\frac{(X-5)}{(X-5)} + \frac{5}{X-5}\frac{(X+5)}{(X+5)} - \frac{10}{X^2-25} = \frac{2X-10+5X+25-10}{X^2-25} = \frac{7X+5}{X^2-25}$$

What if the expression was the same except for the denominator in the second term?

$$\frac{2}{X+5} + \frac{5}{5-X} - \frac{10}{X^2-25}$$

There are three ways of showing that a fraction is negative . $-\frac{1}{2}$ or $\frac{-1}{2}$ or $\frac{1}{-2}$

In the example we can use a double negative since this will still have the same value, just a different form, for +2 = -(-2).

$$\frac{5}{5-X} = \left[-\left(-\frac{5}{5-X}\right)\right] \text{ and } -\frac{5}{5-X} = \frac{5}{-5+X} = \frac{5}{X-5} \text{ so } \frac{5}{5-X} = -\left(\frac{5}{X-5}\right)$$

To transform this equation, we introduce a negative negative so we can use the same denominator as in example 1.

$$\frac{2}{X+5} + \frac{5}{5-X} - \frac{10}{X^2-25} = \frac{2}{X+5} - \frac{5}{X-5} - \frac{10}{X^2-25}$$

Example 2 Combine.

The common denominator is (X+2)(X-2) which is $X^2 - 4$ and $X \neq 2, -2$

$$\frac{X-3}{X-2} + \frac{X+3}{X+2} + \frac{4X+3}{X^2-4} \longrightarrow \frac{(X-3)(X+2)}{(X-2)(X+2)} + \frac{(X+3)(X-2)}{(X+2)(X-2)} + \frac{4X+3}{X^2-4} \longrightarrow \frac{X^2-X-6+X^2+X-6+4X+3}{X^2-4} \longrightarrow \frac{2X^2+4X-9}{X^2-4}$$

Example 3 Combine.

The common denominator is (X+4)(X-3) and X≠ 3,-4

$$\frac{X}{X+4}+\frac{X}{X-3} \longrightarrow \frac{(X)(X-3)}{(X+4)(X-3)}+\frac{(X)(X+4)}{(X-3)(X+4)} \longrightarrow \frac{X^2-3X+X^2+4X}{(X+4)(X-3)} \longrightarrow \frac{2X^2+X}{(X+4)(X-3)}$$

Practice Problems

1) $\frac{4}{X+1}+\frac{7}{X}$ 2) $\frac{X+1}{X+3}+\frac{X-1}{X+2}-\frac{2X}{X^2+5X+6}$ 3) $\frac{X}{X+5}+\frac{3X}{X-2}$

Solutions

The common denominator is (X)(X+1) and X≠ 0,-1

1) $$\frac{4}{X+1}+\frac{7}{X} \longrightarrow \frac{(4)(X)}{(X+1)(X)}+\frac{(7)(X+1)}{(X)(X+1)} \longrightarrow \frac{4X+7X+7}{(X)(X+1)} \longrightarrow \frac{11X+7}{(X)(X+1)}$$

2) $$\frac{X+1}{X+3}+\frac{X-1}{X+2}-\frac{2X}{X^2+5X+6} \longrightarrow \frac{(X+1)(X+2)}{(X+3)(X+2)}+\frac{(X-1)(X+3)}{(X+2)(X+3)}-\frac{2X}{X^2+5X+6} \longrightarrow \frac{X^2+3X+2+X^2+2X-3-2X}{X^2+5X+6} \longrightarrow \frac{2X^2+3X-1}{X^2+5X+6}$$

The common denominator is (X+2)(X+3) which is X^2 + 5X+6 and X≠ -2, -3

3) $$\frac{X}{X+5}+\frac{3X}{X-2} \longrightarrow \frac{(X)(X-2)}{(X+5)(X-2)}+\frac{(3X)(X+5)}{(X-2)(X+5)} \longrightarrow \frac{X^2-2X+3X^2+15X}{(X+5)(X-2)} \longrightarrow \frac{4X^2+13X}{(X+5)(X-2)}$$

The common denominator is (X+5)(X-2) and X≠ -5,2

Another new item is a fraction, or rational expression, divided by another fraction or rational expression. We know that a fraction divided by a fraction is the same as a fraction times its reciprocal. We'll show this, as well as another way to simplify this process by multiplying by 1.

$$\frac{\frac{1}{2}}{\frac{1}{8}} \text{ is the same as: } \frac{1}{2}\div\frac{1}{8}=\frac{1}{2}\times\frac{8}{1}=\frac{4}{1} \text{ or } \frac{\frac{1}{2}}{\frac{1}{8}}\times\frac{\frac{8}{1}}{\frac{8}{1}}=\frac{4}{1}$$

A Fraction Divided by a Fraction In the second method we multiply the denominator 1/8, by its reciprocal 8/1. Then the denominator is 1. But, we can't multiply the denominator by 8/1 without also multiplying the numerator by 8/1, so that we are multiplying the whole fraction by 1, changing its form without affecting its value. Let's do some examples.

Example 4 Simplify.

$$\frac{\frac{2}{X}}{\frac{3}{X+1}} \longrightarrow \frac{\frac{2}{X}}{\frac{3}{X+1}}\cdot\frac{\frac{X+1}{3}}{\frac{X+1}{3}}=\frac{2}{X}\cdot\frac{X+1}{3}=\frac{2X+2}{3X} \quad X\neq 0 \text{ or } -1$$

Example 5 Simplify.

$$\frac{4+\frac{1}{2}}{2-\frac{2}{3}} \longrightarrow \frac{\frac{9}{2}}{\frac{4}{3}} \longrightarrow \frac{\frac{9}{2}}{\frac{4}{3}}\cdot\frac{\frac{3}{4}}{\frac{3}{4}}=\frac{9}{2}\cdot\frac{3}{4}=\frac{27}{8}$$

Example 6 Simplify.

$$\frac{1+\frac{2}{X}}{1+\frac{3}{X+1}} \longrightarrow \frac{\frac{X}{X}+\frac{2}{X}}{\frac{X+1}{X+1}+\frac{3}{X+1}} \longrightarrow \frac{\frac{X+2}{X}}{\frac{X+4}{X+1}} \longrightarrow \frac{\frac{X+2}{X}}{\frac{X+4}{X+1}}\cdot\frac{\frac{X+1}{X+4}}{\frac{X+1}{X+4}}=\frac{X^2+3X+2}{X^2+4X} \quad X\neq 4,0 \text{ or } -1$$

Example 7 Simplify.

$$\frac{\dfrac{X^2+4X+3}{X^2+6X+8}}{\dfrac{X^2-X-2}{X^2+3X-4}} \rightarrow \frac{\dfrac{(X+1)(X+3)}{(X+4)(X+2)}}{\dfrac{(X+1)(X-2)}{(X+4)(X-1)}} \rightarrow \frac{\dfrac{(X+1)(X+3)}{(X+4)(X+2)}}{\dfrac{(X+1)(X-2)}{(X+4)(X-1)}} \cdot \frac{\dfrac{(X+4)(X-1)}{(X+1)(X-2)}}{\dfrac{(X+4)(X-1)}{(X+1)(X-2)}} \rightarrow \frac{\cancel{(X+1)}(X+3)\cdot\cancel{(X+4)}(X-1)}{\cancel{(X+4)}(X+2)\cdot\cancel{(X+1)}(X-2)}$$

$$\rightarrow \frac{(X+3)(X-1)}{(X+2)(X-2)} \rightarrow \frac{X^2+2X-3}{X^2-4} \qquad X \neq -4, -2, 1, -1 \text{ or } 2$$

Practice Problems *Simplify.*

1) $\dfrac{\frac{4}{X}}{\frac{X+1}{2X}}$

2) $\dfrac{2+\frac{1}{4}}{5-\frac{5}{8}}$

3) $\dfrac{1-\frac{5}{A}}{1+\frac{3}{A+2}}$

4) $\dfrac{\frac{X^2-4}{X^2+7X+12}}{\frac{X^2+3X-10}{X^2+6X+9}}$

Solutions

1) $$\frac{\frac{4}{X}}{\frac{X+1}{2X}} \rightarrow \frac{\frac{4}{X}}{\frac{X+1}{2X}} \cdot \frac{\frac{2X}{X+1}}{\frac{2X}{X+1}} = \frac{4}{\cancel{X}} \cdot \frac{2\cancel{X}}{X+1} = \frac{8}{X+1} \qquad X \neq -1 \text{ or } 0$$

2) $$\frac{2+\frac{1}{4}}{5-\frac{5}{8}} \rightarrow \frac{\frac{9}{4}}{\frac{35}{8}} \rightarrow \frac{\frac{9}{4}}{\frac{35}{8}} \cdot \frac{\frac{8}{35}}{\frac{8}{35}} = \frac{9}{\cancel{4}} \cdot \frac{\cancel{8}^{\,2}}{35} = \frac{18}{35}$$

3) $$\frac{1-\frac{5}{A}}{1+\frac{3}{A+2}} \rightarrow \frac{\frac{A}{A}-\frac{5}{A}}{\frac{A+2}{A+2}+\frac{3}{A+2}} \rightarrow \frac{\frac{A-5}{A}}{\frac{A+5}{A+2}} \rightarrow \frac{\frac{A-5}{A}}{\frac{A+5}{A+2}} \cdot \frac{\frac{A+2}{A+5}}{\frac{A+2}{A+5}} = \frac{A^2-3A-10}{A^2+5A} \qquad A \neq -2, 0, -5$$

4) $$\frac{\dfrac{X^2-4}{X^2+7X+12}}{\dfrac{X^2+3X-10}{X^2+6X+9}} \rightarrow \frac{\dfrac{(X+2)(X-2)}{(X+4)(X+3)}}{\dfrac{(X+5)(X-2)}{(X+3)(X+3)}} \rightarrow \frac{\dfrac{(X+2)(X-2)}{(X+4)(X+3)}}{\dfrac{(X+5)(X-2)}{(X+3)(X+3)}} \cdot \frac{\dfrac{(X+3)(X+3)}{(X+5)(X-2)}}{\dfrac{(X+3)(X+3)}{(X+5)(X-2)}} \rightarrow \frac{(X+2)\cancel{(X-2)}\cdot(X+3)\cancel{(X+3)}}{(X+4)\cancel{(X+3)}\cdot(X+5)\cancel{(X-2)}}$$

$$\rightarrow \frac{(X+2)(X+3)}{(X+4)(X+5)} \rightarrow \frac{X^2+5X+6}{X^2+9X+20} \qquad X \neq 2, -3, -4, -5$$