Lesson 11 Completing the Square

In lesson 9, while working on squaring a binomial, we observed the product as the result of squaring the first term, adding (or subtracting) two times the middle term, and squaring the last term. We also learned to recognize the square root of a binomial squared. Let's begin by reviewing this and then finding out what is needed to make a perfect square.

Example 1 Square the binomial.

$(X + 7)^2$ The first term squared is X^2 and the last term squared is 49. The middle term is the first term times the second term times 2, $(7X) \cdot 2 = 14X$. $X^2 + 14X + 49$

Example 2 Square the binomial.

$(X - 6)^2$ The first term squared is X^2 and the last term squared is 36. The middle term is the first term times the second term times 2, $(-6X) \cdot 2 = -12X$. $X^2 - 12X + 36$

Example 3 Square the binomial.

$(X + 2/3)^2$ The first term squared is X^2 and the last term squared is 4/9. The middle term is the first term times the second term times 2, $(2/3X) \cdot 2 = 4/3X$. $X^2 + 4/3X + 4/9$

What is needed to make a perfect square if you only have the first and middle terms? Looking at the examples we can see that the last term is half of the middle term squared. So if you are given the first and middle, take half of the coefficient of the middle term and square it.

Example 4 Complete the square by finding the last term.

$X^2 + 26X +$ _____ Half of 26 is 13, which when squared is 169. $X^2 + 26X + 169 = (X + 13)^2$

Example 5 Complete the square by finding the last term.

$X^2 - 2X +$ _____ Half of (-2) is (-1), which when squared is 1. $X^2 - 2X + 1 = (X - 1)^2$

Example 6 Complete the square by finding the last term.

$X^2 + 5X +$ _____ Half of 5 is 5/2, which when squared is 25/4. $X^2 + 5X + 25/4 = (X + 5/2)^2$

Example 7 Complete the square by finding the last term.

$X^2 + 3/5X +$ _____ Half of 3/5 is 3/10, which when squared is 9/100. $X^2 + 3/5X + 9/100 = (X + 3/10)^2$

Practice Problems *Complete the square by finding the last term.*

1) $X^2 + 18X +$ _____ 2) $X^2 - 4X +$ _____ 3) $X^2 + 7X +$ _____ 4) $X^2 - 11X +$ _____

5) $X^2 + X +$ _____ 6) $X^2 - 1/2X +$ _____ 7) $X^2 + 3/4X +$ ____ 8) $X^2 - 5/3X +$ _____

Solutions

1) $X^2 + 18X + 81$ 2) $X^2 - 4X + 4$ 3) $X^2 + 7X + 49/4$ 4) $X^2 - 11X + 121/4$

5) $X^2 + X + 1/4$ 6) $X^2 - 1/2X + 1/16$ 7) $X^2 + 3/4X + 9/64$ 8) $X^2 - 5/3X + 25/36$

What is needed to make a perfect square if you only have the first and last terms? Looking at the examples we can see that the last term is half of the coefficient of the middle term squared. So if you are given the last term, take the square root of it and double it to find the coefficient of the middle term.

Example 8 Complete the square by finding the middle term.

$X^2 + \_\_\_\_ + 100$ The square root of 100 is 10, when doubled, 20. $X^2 + 20X + 100 = (X + 10)^2$

Example 9 Complete the square by finding the middle term.

$X^2 - \_\_\_\_ + 64$ The square root of 64 is 8, when doubled, 16. The negative sign makes the root (-8). $X^2 - 16X + 64 = (X - 8)^2$

Example 10 Complete the square by finding the middle term.

$X^2 + \_\_\_\_ + 1/9$ The square root of 1/9 is 1/3, when doubled, 2/3. $X^2 + 2/3X + 1/9 = (X + 1/3)^2$

Practice Problems *Complete the square by finding the middle term.*

1) $X^2 + \_\_\_\_ + 144$ 2) $X^2 - \_\_\_\_ + 169$ 3) $X^2 + \_\_\_\_ + 121$ 4) $X^2 - \_\_\_\_ + 49$

5) $X^2 + \_\_\_\_ + 36$ 6) $X^2 - \_\_\_\_ + 16/25$ 7) $X^2 - \_\_\_\_ + 9/64$ 8) $X^2 + \_\_\_\_ + 1/4$

Solutions

1) $X^2 + 24X + 144$ $(X+12)^2$

2) $X^2 - 26X + 169$ $(X-13)^2$

3) $X^2 + 22X + 121$ $(X+11)^2$

4) $X^2 - 14X + 49$ $(X-7)^2$

5) $X^2 + 12X + 36$ $(X+6)^2$

6) $X^2 - 8/5X + 16/25$ $(X-4/5)^2$

7) $X^2 - 3/4X + 9/64$ $(X-3/8)^2$

8) $X^2 + X + 1/4$ $(X+1/2)^2$

Now that you are an expert at completing the square, we are going to apply this knowledge to solving equations that we want to factor, but which have no whole number solutions.

Example 11 Solve for X by completing the square. $X^2 + 8X - 10 = 0$

Step 1: Add 10 to both sides. $X^2 + 8X = 10$

Step 2: Complete the square and add to both sides. $X^2 + 8X + 16 = 10 + 16$

Step 3: Rewrite as a square. $(X+4)^2 = 26$

Step 4: Take the square root of both sides. $\sqrt{(X+4)^2} = \sqrt{26}$

Step 5: Solve for X. $X+4 = \sqrt{26}$

$X = -4 \pm \sqrt{26}$

The solutions, or roots, are $-4 + \sqrt{26}$ and $-4 - \sqrt{26}$. We put these into the original equation to see if they check. They do.

$(-4 + \sqrt{26})^2 + 8(-4 + \sqrt{26}) - 10 = 0$

$16 - 8\sqrt{26} + 26 - 32 + 8\sqrt{26} - 10 = 0$

$42 - 32 - 10 = 0$

$(-4 - \sqrt{26})^2 + 8(-4 - \sqrt{26}) - 10 = 0$

$16 + 8\sqrt{26} + 26 - 32 - 8\sqrt{26} - 10 = 0$

$42 - 32 - 10 = 0$

Example 12 Solve for X by completing the square. $X^2 - 6X + 4 = 0$

Step 1: Add -4 to both sides. $X^2 - 6X = -4$

Step 2: Complete the square and add to both sides. $X^2 - 6X + 9 = -4 + 9$

Step 3: Rewrite as a square. $(X - 3)^2 = 5$

Step 4: Take the square root of both sides. $\sqrt{(X-3)^2} = \pm\sqrt{5}$

Step 5: Solve for X. $X - 3 = \pm\sqrt{5}$

$X = 3 \pm \sqrt{5}$

The solutions, or roots, are $3 + \sqrt{5}$ and $3 - \sqrt{5}$. We put these into the original equation to see if they check. They do.

$(3 + \sqrt{5})^2 - 6(3 + \sqrt{5}) + 4 = 0$

$9 + 6\sqrt{5} + 5 - 18 - 6\sqrt{5} + 4 = 0$

$14 - 18 + 4 = 0$

$(3 - \sqrt{5})^2 - 6(3 - \sqrt{5}) + 4 = 0$

$9 - 6\sqrt{5} + 5 - 18 + 6\sqrt{5} + 4 = 0$

$14 - 18 + 4 = 0$

Example 13 Solve for X by completing the square. If there is a coefficient before the first term, divide through by that number. It is easier to complete the square when the coefficient of X^2 is 1.

$2X^2 + 10X + 4 = 0$

Step A: Divide everything by 2. $X^2 + 5X + 2 = 0$

Step 1: Add -2 to both sides. $X^2 + 5X = -2$

Step 2: Complete the square and add to both sides. $X^2 + 5X + \frac{25}{4} = \frac{-8}{4} + \frac{25}{4}$

Step 3: Rewrite as a square. $(X + \frac{5}{2})^2 = \frac{17}{4}$

Step 4: Take the square root of both sides. $\sqrt{(X + \frac{5}{2})^2} = \pm\sqrt{\frac{17}{4}}$

Step 5: Solve for X. $X + \frac{5}{2} = \pm\sqrt{\frac{17}{4}}$

$X = -\frac{5}{2} \pm \frac{\sqrt{17}}{2}$

The solutions, or roots, are $(-5 + \sqrt{17})/2$ and $(-5 - \sqrt{17})/2$. We put these into the original equation and they check.

$(\frac{-5}{2} + \frac{\sqrt{17}}{2})^2 + 5(\frac{-5}{2} + \frac{\sqrt{17}}{2}) + 2 = 0$

$\frac{25}{4} - \frac{10\sqrt{17}}{4} + \frac{17}{4} - \frac{25}{2} + \frac{5\sqrt{17}}{2} + 2 = 0$

$\frac{25}{4} - \frac{5\sqrt{17}}{2} + \frac{17}{4} - \frac{50}{4} + \frac{5\sqrt{17}}{2} + \frac{8}{4} = 0$

$0 = 0$

$(\frac{-5}{2} - \frac{\sqrt{17}}{2})^2 + 5(\frac{-5}{2} - \frac{\sqrt{17}}{2}) + 2 = 0$

$\frac{25}{4} + \frac{10\sqrt{17}}{4} + \frac{17}{4} - \frac{25}{2} - \frac{5\sqrt{17}}{2} + 2 = 0$

$\frac{25}{4} + \frac{5\sqrt{17}}{2} + \frac{17}{4} - \frac{50}{4} - \frac{5\sqrt{17}}{2} + \frac{8}{4} = 0$

$0 = 0$

Practice Problems Solve for X by completing the square and check your work.

1) $X^2 + 8X + 1 = 0$

2) $X^2 - 6X - 2 = 0$

3) $2X^2 + 4X + 10 = 0$

4) $3X^2 - 36X - 36 = 0$

5) $X^2 + 3X + 1 = 0$

6) $X^2 - 7X + 5 = 0$

Solutions

1) $X^2 + 8X + 1 = 0$

$X^2 + 8X = -1$

$X^2 + 8X + 16 = -1 + 16$

$(X+4)^2 = 15$

$\sqrt{(X+4)^2} = \sqrt{15}$

$X + 4 = \pm\sqrt{15}$

$X = -4 \pm \sqrt{15}$

$(-4 + \sqrt{15})^2 + 8(-4 + \sqrt{15}) + 1 = 0$

$16 - 8 + \sqrt{15} + 15 - 32 + 8\sqrt{15} + 1 = 0$

$31 - 32 + 1 = 0$

$(-4 - \sqrt{15})^2 + 8(-4 - \sqrt{15}) + 1 = 0$

$16 + 8\sqrt{15} + 15 - 32 - 8\sqrt{15} + 1 = 0$

$31 - 32 + 1 = 0$

3) $2X^2 + 4X + 10 = 0$

$X^2 + 2X + 5 = 0$

$X^2 + 2X = -5$

$X^2 + 2X + 1 = -5 + 1$

$(X+1)^2 = -4$

$\sqrt{(X+1)^2} = \sqrt{-4}$

$X + 1 = \pm\sqrt{-4}$

$X = -1 \pm 2i$

$(-1 + 2i)^2 + 2(-1 + 2i) + 5 = 0$

$1 - 4i - 4 - 2 + 4i + 5 = 0$

$6 - 6 = 0$

$(-1 - 2i)^2 + 2(-1 - 2i) + 5 = 0$

$1 + 4i - 4 - 2 - 4i + 5 = 0$

$6 - 6 = 0$

5) $X^2 + 3X + 1 = 0$

$X^2 + 3X = -1$

$X^2 + 3X + 9/4 = -1 + 9/4$

$(X + 3/2)^2 = 5/4$

$\sqrt{(X + 3/2)^2} = \sqrt{5/4}$

$X + 3/2 = \pm\sqrt{5/4}$

$X = \frac{-3}{2} \pm \frac{\sqrt{5}}{2}$

$(\frac{-3}{2} + \frac{\sqrt{5}}{2})^2 + 3(\frac{-3}{2} + \frac{\sqrt{5}}{2}) + 1 = 0$

$\frac{9}{4} - \frac{6\sqrt{5}}{4} + \frac{5}{4} - \frac{9}{2} + \frac{3\sqrt{5}}{2} + 1 = 0$

$\frac{9}{4} - \frac{3\sqrt{5}}{2} + \frac{5}{4} - \frac{18}{4} + \frac{3\sqrt{5}}{2} + \frac{4}{4} = 0$

$(\frac{-3}{2} - \frac{\sqrt{5}}{2})^2 + 3(\frac{-3}{2} - \frac{\sqrt{5}}{2}) + 1 = 0$

$\frac{9}{4} + \frac{6\sqrt{5}}{4} + \frac{5}{4} - \frac{9}{2} - \frac{3\sqrt{5}}{2} + 1 = 0$

$\frac{9}{4} + \frac{3\sqrt{5}}{2} + \frac{5}{4} - \frac{18}{4} - \frac{3\sqrt{5}}{2} + \frac{4}{4} = 0$

2) $X^2 - 6X - 2 = 0$

$X^2 - 6X = 2$

$X^2 - 6X + 9 = 2 + 9$

$(X-3)^2 = 11$

$\sqrt{(X-3)^2} = \sqrt{11}$

$X - 3 = \pm\sqrt{11}$

$X = 3 \pm \sqrt{11}$

$(3 + \sqrt{11})^2 - 6(3 + \sqrt{11}) - 2 = 0$

$9 + 6\sqrt{11} + 11 - 18 - 6\sqrt{11} - 2 = 0$

$20 - 18 - 2 = 0$

$(3 - \sqrt{11})^2 - 6(3 - \sqrt{11}) - 2 = 0$

$9 - 6\sqrt{11} + 11 - 18 + 6\sqrt{11} - 2 = 0$

$20 - 18 - 2 = 0$

4) $3X^2 - 36X - 36 = 0$

$X^2 - 12X - 12 = 0$

$X^2 - 12X = 12$

$X^2 - 12X + 36 = 12 + 36$

$(X-6)^2 = 48$

$\sqrt{(X-6)^2} = \sqrt{48}$

$X - 6 = \pm\sqrt{48}$

$X = 6 \pm 4\sqrt{3}$

$(6 + 4\sqrt{3})^2 - 12(6 + 4\sqrt{3}) - 12 = 0$

$36 + 48\sqrt{3} + 16 \cdot 3 - 72 - 48\sqrt{3} - 12 = 0$

$84 - 84 = 0$

$(6 - 4\sqrt{3})^2 - 12(6 - 4\sqrt{3}) - 12 = 0$

$36 - 48\sqrt{3} + 16 \cdot 3 - 72 + 48\sqrt{3} - 12 = 0$

$84 - 84 = 0$

6) $X^2 - 7X + 5 = 0$

$X^2 - 7X = -5$

$X^2 - 7X + 49/4 = -5 + 49/4$

$(X - 7/2)^2 = 29/4$

$\sqrt{(X - 7/2)^2} = \sqrt{29/4}$

$X - 7/2 = \pm\sqrt{29/4}$

$X = 7/2 \pm \sqrt{29}/2$

$(\frac{7}{2} + \frac{\sqrt{29}}{2})^2 - 7(\frac{7}{2} + \frac{\sqrt{29}}{2}) + 5 = 0$

$\frac{49}{4} + \frac{14\sqrt{29}}{4} + \frac{29}{4} - \frac{49}{2} - \frac{7\sqrt{29}}{2} + 5 = 0$

$\frac{49}{4} + \frac{7\sqrt{29}}{2} + \frac{29}{4} - \frac{98}{4} - \frac{7\sqrt{29}}{2} + \frac{20}{4} = 0$

$(\frac{7}{2} - \frac{\sqrt{29}}{2})^2 - 7(\frac{7}{2} + \frac{\sqrt{29}}{2}) + 5 = 0$

$\frac{49}{4} - \frac{14\sqrt{29}}{4} + \frac{29}{4} - \frac{49}{2} + \frac{7\sqrt{29}}{2} + 5 = 0$

$\frac{49}{4} + \frac{7\sqrt{29}}{2} + \frac{29}{4} - \frac{98}{4} + \frac{7\sqrt{29}}{2} + \frac{20}{4} = 0$